Distribution Free Goodness-of-Fit Tests for Linear Processes*

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Abstract

This article proposes a class of goodness-of-fit tests for the autocorrelation function of a time series process, including those exhibiting long-range dependence. Test statistics for composite hypotheses are functionals of a (approximated) martingale transformation of the Bartlett's T_p-*process* with estimated parameters, which converges in distribution to the standard Brownian Motion under the null hypothesis. We discuss tests of different nature such as omnibus, directional and Portmanteau-type tests. A Monte Carlo study illustrates the performance of the different tests in practice.

Keywords: Nonparametric model checking; spectral distribution; linear processes; martingale decomposition; local alternatives; omnibus, smooth and directional tests; long-range alternatives.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Let f be the spectral density function of a second order stationary time series process $\{X(t)\}_{t\in\mathbb{Z}}$ with mean μ and covariance function

$$Cov\left(X\left(j\right), X\left(0\right)\right) = \int_{-\pi}^{\pi} f\left(\lambda\right)\cos\left(\lambda j\right) d\lambda; \quad j = 0, \pm 1, \pm 2, \ldots.$$

We shall assume that $\{X(t)\}_{t\in\mathbb{Z}}$ admits the Wold's representation

$$X\left(t\right) = \mu + \sum_{j=0}^{\infty} a\left(j\right)\varepsilon\left(t-j\right), \text{ with } a\left(0\right) = 1 \text{ and } \sum_{j=0}^{\infty} a^2\left(j\right) < \infty, \tag{1}$$

for some sequence $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}\left(\varepsilon\left(t\right)\right) = 0$ and $\mathbb{E}\left(\varepsilon\left(0\right)\varepsilon\left(t\right)\right) = \sigma^2$ if $t = 0$; and $= 0$ for all $t \neq 0$. Under (1), the spectral density function of $\{X(t)\}_{t\in\mathbb{Z}}$ can be factorized as

$$f\left(\lambda\right) = \frac{\sigma^2}{2\pi} h\left(\lambda\right), \ \lambda \in [0, \pi],$$

with $h\left(\lambda\right) := \left|\sum_{j=0}^{\infty} a\left(j\right) e^{ij\lambda}\right|^2$.

Let

$$\mathcal{H} = \left\{ h_\theta : \int_0^{\pi} \log h_\theta\left(\lambda\right) d\lambda = 0, \ \theta \in \Theta \right\}, \tag{2}$$

where $\Theta \subset \mathbb{R}^p$ is a compact parameter space. Much of the existing time series literature is concerned with parametric estimation and testing, assuming that h belongs to $\mathcal{H}$, i.e. $h = h_{\theta_0}$ for some $\theta_0 \in \Theta$, because the parameter θ_0 and the functional form of h_θ summarize the autocorrelation structure of $\{X(t)\}_{t\in\mathbb{Z}}$. Notice that $h \in \mathcal{H}$ in (2) guarantees that $a(0) = 1$ in (1) and $\sigma^2 = \min_{\theta\in\Theta} 2\int_0^{\pi} f\left(\lambda\right)/h_\theta\left(\lambda\right) d\lambda$. For our purposes, σ^2 can be considered a nuisance parameter, as is also the mean μ.

Classical parameterizations that accommodate alternative models are the ARMA, ARFIMA, fractional noise or Bloomfield's (1973) exponential models (see Robinson, 1994 for definitions). For instance, in an ARFIMA specification, $\mathcal{H}$ consists of all functions indexed by a parameter vector $\theta = \left(d, \eta', \delta'\right)'$, where $\theta \in \Theta \subset (-1/2, 1/2) \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, of the form

$$h_\theta\left(\lambda\right) = \frac{1}{\left|1 - e^{i\lambda}\right|^{2d}} \left|\frac{\Xi_\eta\left(e^{i\lambda}\right)}{\Phi_\delta\left(e^{i\lambda}\right)}\right|^2, \ \lambda \in [0, \pi], \tag{3}$$

such that Ξ_η and Φ_δ are the moving average and autoregressive polynomials of orders p_1 and p_2, respectively, with no common roots, all lying outside the unit circle.

Before statistical inference on the true value θ_0 is made, one needs to test the hypothesis $H_0 : h \in \mathcal{H}$, which can be equivalently stated as

$$H_0 : \frac{G_{\theta_0}(\lambda)}{G_{\theta_0}(\pi)} = \frac{\lambda}{\pi} \text{ for all } \lambda \in [0, \pi] \text{ and some } \theta_0 \in \Theta, \tag{4}$$

where

$$G_\theta(\lambda) := 2\int_0^\lambda \frac{f(\bar{\lambda})}{h_\theta(\bar{\lambda})} d\bar{\lambda},\ \lambda \in [0, \pi].$$

Under H_0, G_{θ_0} is the spectral distribution function of the innovation process $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$ and $G_{\theta_0}(\pi) = \sigma^2$.

Given a record $\{X(t)\}_{t=1}^T$ and a consistent estimator θ_T of θ_0 under H_0, a natural estimator of G_{θ_0} is defined as $G_{\theta_T,T}(\lambda)$, where

$$G_{\theta,T}(\lambda) := \frac{2\pi}{\tilde{T}} \sum_{j=1}^{[\tilde{T}\lambda/\pi]} \frac{I_X(\lambda_j)}{h_\theta(\lambda_j)},\ \lambda \in [0, \pi]. \tag{5}$$

Here $\tilde{T} = [T/2]$, $[z]$ being the integer part of z, and for a generic time series process $\{V(t)\}_{t\in\mathbb{Z}}$,

$$I_V(\lambda_j) := \frac{1}{2\pi T}\left|\sum_{t=1}^T V(t) e^{it\lambda_j}\right|^2,\ j = 1, ..., \tilde{T},$$

denotes the periodogram of $\{V(t)\}_{t=1}^T$ evaluated at the Fourier frequency $\lambda_j = 2\pi j/T$ for positive integers j.

The formulation of H_0 in (4) suggests to use Bartlett's $T_p - process$ as a basis for testing H_0. The $T_p - process$ is defined as

$$\alpha_{\theta,T}(\lambda) := \tilde{T}^{1/2}\left[\frac{G_{\theta,T}(\lambda)}{G_{\theta,T}(\pi)} - \frac{\lambda}{\pi}\right],\ \lambda \in [0, \pi].$$

Notice that $\alpha_{\theta,T}$ is scale invariant and that, for $j \neq 0, \mathrm{mod}(T)$, $I_V(\lambda_j)$ is mean invariant, so omission of $j = 0$ in the definition of $G_{\theta,T}$ entails mean correction. That is, $\alpha_{\theta,T}$ is independent of both μ and σ^2.

Under short-range dependence and H_0, we have that

$$\max_{1\le j\le\tilde{T}} \mathbb{E}\left|\frac{I_X(\lambda_j)}{h_{\theta_0}(\lambda_j)} - I_\varepsilon(\lambda_j)\right| = o(1),$$

see Brockwell and Davis (1991, Theorem 10.3.1, p. 346). So, it is expected that $\alpha_{\theta_0,T}$ will be asymptotically equivalent to Bartlett's $U_p - process$ for $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$,

$$\alpha_T^0(\lambda) := \tilde{T}^{1/2}\left[\frac{G_T^0(\lambda)}{G_T^0(\pi)} - \frac{\lambda}{\pi}\right],\ \lambda \in [0, \pi],$$

with

$$G_T^0(\lambda) := \frac{2\pi}{\tilde{T}} \sum_{j=1}^{[\tilde{T}\lambda/\pi]} I_\varepsilon(\lambda_j), \ \lambda \in [0,\pi].$$

In fact, under suitable regularity conditions, we shall show below that the aforementioned equivalence holds also true under long-range dependence. Observe that the $U_p - process$ α_T^0 and the $T_p - process$ $\alpha_{\theta_0,T}$ are identical when $\{X(t)\}_{t\in\mathbb{Z}}$ is a white noise process.

The $U_p - process$ α_T^0 is useful for testing simple hypotheses when the innovations $\{\varepsilon(t)\}_{t=1}^T$ can be easily computed, as is the case when $\{X(t)\}_{t\in\mathbb{Z}}$ is an AR model. However, there are many other models of interest whose innovations $\{\varepsilon(t)\}_{t=1}^T$ cannot be directly computed, e.g. Bloomfield's exponential model, or difficult to obtain, like in models exhibiting long-range dependence, such as ARFIMA models. In those cases, it appears computationally much simpler to use $\alpha_{\theta_0,T}$ for testing simple hypotheses.

The empirical processes α_T^0 and $\alpha_{\theta,T}$, with fixed θ, are random elements in $D[0,\pi]$, the space of right continuous functions on $[0,\pi]$ with left hand side limits, the càdlàg space. The functional space $D[0,\pi]$ is endowed with the Skorohod's metric (see e.g. Billingsley, 1968) and convergence in distribution in the corresponding topology will be denoted by "$\Rightarrow$ ".

Under suitable regularity conditions on $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$, it is well known that

$$\alpha_T^0 \Rightarrow B_\pi^1, \tag{6}$$

where B_π^1 is the standardized tied down Brownian motion at π. In terms of the standard Brownian motion B on $[0,1]$, B_π^1 can be represented as

$$B_\pi^1(\lambda) = B\left(\frac{\lambda}{\pi}\right) - \frac{\lambda}{\pi} B(1), \ \lambda \in [0,\pi].$$

Grenander and Rosenblatt (1957) proved (6) assuming that $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$ is a sequence of independent and identically distributed (*iid*) random variables with eight bounded moments. The *iid* condition was relaxed by Dahlhaus (1985), who assumed that $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$ behaves as a martingale difference, but still assuming eight bounded moments. Recently Klüppelberg and Mikosch (1996) proved (6) under *iid* $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$, but assuming only four bounded moments. The *iid* requirement is relaxed by the following assumption.

A1 The innovation process $\{\varepsilon(t)\}_{t\in\mathbb{Z}}$ satisfies that $\mathbb{E}(\varepsilon(t)^r | \mathcal{F}_{t-1}) = \mu_r$ with μ_r constant ($\mu_1 = 0$ and $\mu_2 = \sigma^2$) for $r = 1, \ldots, 4$ and all $t = 0, \pm 1, \ldots$, where $\mathcal{F}_t$ is the sigma algebra generated by $\{\varepsilon(s), s \le t\}$.

Assumption $A1$ appears a minimal requirement to establish a functional central limit theorem for α_T^0, due to the quadratic nature of the periodogram.

To establish the asymptotic equivalence between $\alpha_{\theta_0,T}$ and α_T^0, we introduce the following smoothness assumptions on h.

A2 (a) h is a positive and continuously differentiable function on $(0,\pi]$;

(b) $|\partial \log h(\lambda)/\partial\lambda| = O\left(\lambda^{-1}\right)$ as $\lambda \to 0+$.

This condition is very general and allows for a possible singularity of h at $\lambda = 0$. It holds for models exhibiting long-range dependence, like ARFIMA(p_2, d, p_1) models with $d \neq 0$, as can easily be checked using (3) and that $\left|1 - e^{i\lambda}\right| = |2\sin(\lambda/2)|$.

Theorem 1 *Assuming* $A1$ *and* $A2$, *under* H_0, *(6) holds and*

$$\sup_{\lambda\in[0,\pi]} \left|\alpha_{\theta_0,T}(\lambda) - \alpha_T^0(\lambda)\right| = o_p(1).$$

We can relax the location of the possible singularity in h at any other frequency $\lambda \neq 0$, as in Hosoya (1997) or, more recently, Giraitis, Hidalgo and Robinson (2001), or even allow for more than one singularity. However, for notational simplicity, we have taken the singularity, if any, at $\lambda = 0$. If the location of the singularity were at $\lambda^0 \neq 0$, then $A2$ would be modified to

A2' (a) h is a positive and continuously differentiable function on $\left[0,\lambda^0\right) \cup \left(\lambda^0,\pi\right]$;

(b) $|\partial \log h(\lambda)/\partial\lambda| = O\left(\left|\lambda - \lambda^0\right|^{-1}\right)$ as $\lambda \to \lambda^0$.

We now comment on the results of Theorem 1. This theorem indicates that $\alpha_{\theta_0,T}$ is asymptotically pivotal. One consequence is that critical regions of tests based on a continuous functional $\varphi : D[0,\pi] \mapsto \mathbb{R}$ can be easily obtained. Different functionals φ lead to tests with different power properties. Among them are omnibus, directional and/or Portmanteau-type tests. For example, classical functionals which lead to omnibus tests are the Kolmogorov-Smirnov ($\varphi(g) = \sup_{\lambda\in[0,\pi]} |g(\lambda)|$) and the Cramér-von Mises ($\varphi(g) = \pi^{-1}\int_0^\pi g(\lambda)^2 d\lambda$), whereas Portmanteau tests, defined as weighted sums of squared estimated autocorrelations of the innovations, and directional tests are obtained by choosing an appropriate functional φ, see Section 3 for details.

On the other hand, in practical situations the parameters θ_0 are not known and, thus, they have to be replaced by some estimate θ_T. In this situation, as Theorem 2 below shows, the $T_p - process$ is no longer asymptotically pivotal, and hence the aforementioned tests are not useful for practical purposes. The unknown critical values of functionals of the $T_p - process$ with estimated parameters can be approximated with the assistance of bootstrap methods. This approach has been proposed by Chen and Romano (2000) or Hainz and Dahlhaus (2000) for short-range models using the $U_p - process$ and by Delgado and Hidalgo (2000), who allow also long-range dependence models using the $T_p - process$. Alternatively, asymptotically distribution free tests can be obtained by introducing a tuning parameter that must behave in some required way as the sample size increases. Among them, the most popular one is the Portmanteau test, although it has only been justified for testing short-range models. Box and Pierce (1970) showed that the partial sum of the residuals squared autocorrelations of a stationary ARMA process is approximately chi-squared distributed assuming that the number of autocorrelations considered diverges to infinity with the sample size at an appropriate rate. A different approach, in the spirit of Durbin, Knott and Taylor (1976) for the classical empirical process, is that in Anderson (1997), who proposed to approximate the critical values of the Cramér-von Mises tests for a stationary AR model. The method considers a truncated version of the spectral representation of $\alpha_{\theta_T,T}$ with estimated orthogonal components. The number of estimated orthogonal components must suitably increase with the sample size. A similar idea was proposed by Velilla (1996) for ARMA models. Finally, another alternative uses the distance between a smooth estimator of the spectral density function and its parametric estimator under H_0. This approach provides asymptotically distribution free tests for short-range models assuming a suitable behavior of the smoothing parameter as the sample size diverges, see e.g. Prewitt (1998) and Paparoditis (2000). However, the final outcome of all these tests depends on the arbitrary choice of the tuning/smoothing parameters for which no relevant theory is available.

This article solves some limitations of existing asymptotically pivotal tests, only justified under short-range dependence, by considering an asymptotically pivotal transformation of $\alpha_{\theta_T,T}$ related to the cusum of recursive residuals proposed by Brown, Durbin and Evans (1975). We show that our testing procedure is valid under long-range specifications. In the next section we provide regularity conditions for the weak convergence of $\alpha_{\theta_T,T}$ and its asymptotically distribution free transformation. In Section 3, we discuss the behavior of tests of very different nature -omnibus, directional and smooth/Portmanteau- under local alternatives converging to the null at the rate $T^{-1/2}$. Section 4 reports the results of a small Monte Carlo experiment. Some final remarks are placed in Section 5. Section 6 provides a Lemmata with some auxiliary results, which are employed to prove, in Section 7, the main results of the paper.

2. TESTS BASED ON A MARTINGALE TRANSFORMATION OF THE $\mathbf{T}_P$ -PROCESS WITH ESTIMATED PARAMETERS

A popular estimator of θ_0 is the Whittle estimator

$$\theta_T := \arg\min_{\theta\in\Theta} G_{\theta,T}(\pi), \tag{7}$$

with $G_{\theta,T}$ defined in (5). Let us define

$$\phi_\theta(\lambda) := \frac{\partial}{\partial\theta}\log h_\theta(\lambda); \quad S_T := \frac{1}{\tilde{T}}\sum_{j=1}^{\tilde{T}}\phi_{\theta_0}(\lambda_j)\,\phi'_{\theta_0}(\lambda_j)$$

and introduce the following assumptions,

A3 (a) ϕ_{θ_0} is a continuously differentiable function on $(0,\pi]$; (b) $\left\|\partial\phi_{\theta_0}(\lambda)/\partial\lambda\right\| = O(1/\lambda)$ as $\lambda \to 0+$; and for some $0<\delta<1$ and all $\lambda \in (0,\pi]$, there exists a $K<\infty$ such that (c) $\sup_{\{\theta:\|\theta-\theta_0\|\le\delta\}} \|\phi_\theta(\lambda)\| \le K|\log\lambda|$; (d)

$$\sup_{\{\theta:\|\theta-\theta_0\|\le\delta/2\}} \frac{1}{\|\theta-\theta_0\|^2}\left|\frac{h_{\theta_0}(\lambda)}{h_\theta(\lambda)} - 1 + \phi'_{\theta_0}(\lambda)(\theta-\theta_0)\right| \le \frac{K}{\lambda^\delta}\log^2\lambda;$$

and (e) $\Sigma_{\theta_0} := \pi^{-1}\int_0^\pi \phi_{\theta_0}(\lambda)\phi'_{\theta_0}(\lambda)\,d\lambda$ is positive definite.

These assumptions are standard when analyzing the asymptotic distribution of the Whittle estimator θ_T and they are satisfied for all parametric linear processes used in practice. Standard ARMA models satisfy a stronger condition, replacing the upper bounds in $A3(c)$ and (d) by a constant independent of λ. It can be easily shown that $A3$ is satisfied for ARFIMA models. Note that $A3\,(e)$ and Lemma 1 in Section 6 imply that S_T is positive definite for T large enough.

A4 The estimator in (7) satisfies the asymptotic linearization

$$\widetilde{T}^{1/2}(\theta_T-\theta_0) = S_T^{-1}\int_0^\pi \phi_{\theta_0}(\lambda)\,\alpha_{\theta_0,T}(d\lambda) + o_p(1). \tag{8}$$

The expansion (8), in Assumption $A4$, is satisfied under $A1-A3$ and additional standard identification conditions, see Hannan (1973), Giraitis and Surgailis (1990), or Velasco and Robinson (2000) for a later reference.

Define

$$\alpha_\infty(\lambda) := B_\pi^1(\lambda) - \left(\frac{1}{\pi}\int_0^\lambda \phi'_{\theta_0}(\bar\lambda)\,d\bar\lambda\right)\Sigma_{\theta_0}^{-1}\int_0^\pi \phi_{\theta_0}(\bar\lambda)\,B_\pi^1(d\bar\lambda).$$

Theorem 2 *Under H_0 and assuming $A1-A4$, uniformly in $\lambda\in[0,\pi]$,*

(a) $$\alpha_{\theta_T,T}(\lambda) = \alpha_T^0(\lambda) - \left(\frac{1}{\widetilde{T}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \phi'_{\theta_0}(\lambda_j)\right) S_T^{-1}\int_0^\pi \phi_{\theta_0}(\bar\lambda)\,\alpha_T^0(d\bar\lambda) + o_p(1);$$

(b) $\alpha_{\theta_T,T} \Rightarrow \alpha_\infty$.

Theorem 2 indicates that the asymptotic critical values of tests based on $\alpha_{\theta_T,T}$ cannot be tabulated. However, we can use a transformation of $\alpha_{\theta_T,T}$ that converges in distribution to the standard Brownian motion. To this end, it is of interest to realize that Theorem 2 (a) provides an asymptotic representation of $\alpha_{\theta_T,T}$ as a scaled cumulative sum (cusum) of the least squares residuals in an artificial regression model. For that purpose, observe that by (2), and using the fact that ϕ_{θ_0} is integrable (A3 (c)),

$$\int_0^\pi \phi_{\theta_0}(\lambda)\, d\lambda = 0. \tag{9}$$

Now, because Lemma 1 in Section 6 with $\zeta(\lambda) = \phi_{\theta_0}(\lambda)$ and (9) imply that $\left\| \sum_{k=1}^{\tilde{T}} \phi_{\theta_0}(\lambda_k) \right\| = O(\log T)$, the uniform asymptotic expansion in Theorem 2 (a) indicates that

$$\sup_{\lambda\in[0,\pi]} \left| \alpha_{\theta_T,T}(\lambda) - \frac{2\pi}{G_T^0(\pi)} \frac{1}{\tilde{T}^{1/2}} \sum_{j=1}^{[\tilde{T}\lambda/\pi]} u_T(j) \right| = o_p(1),$$

where

$$u_T(j) = I_\varepsilon(\lambda_j) - \gamma'_{\theta_0}(\lambda_j) \left[\sum_{k=1}^{\tilde{T}} \gamma_{\theta_0}(\lambda_k)\gamma'_{\theta_0}(\lambda_k) \right]^{-1} \sum_{k=1}^{\tilde{T}} \gamma_{\theta_0}(\lambda_k) I_\varepsilon(\lambda_k),\ j = 1, ..., \tilde{T}$$

are the least squares residuals in an artificial regression model with dependent variable $I_\varepsilon(\lambda_j)$ and a vector of explanatory variables $\gamma_{\theta_0}(\lambda_j) := \left(1, \phi'_{\theta_0}(\lambda_j)\right)'$. This fact suggests to employ the cusum of recursive residuals for constructing asymptotically pivotal tests, as they were proposed by Brown, Durbin and Evans (1975), see also Sen (1982).

Let us define

$$A_{\theta,T}(j) := \frac{1}{\tilde{T}} \sum_{k=j+1}^{\tilde{T}} \gamma_\theta(\lambda_k)\gamma'_\theta(\lambda_k),$$

and assume that

A5 $A_{\theta_0,T}\left(\bar{T}\right)$ is non singular for $\bar{T} = \tilde{T} - p - 1$.

The (scaled) cusum of forward recursive least squares residuals is defined as

$$\beta_T^0(\lambda) := \frac{2\pi}{G_T^0(\pi)} \frac{1}{\tilde{T}^{1/2}} \sum_{j=1}^{[\bar{T}\lambda/\pi]} e_T(j),\ \lambda \in [0,\pi],$$

where

$$e_T(j) := I_\varepsilon(\lambda_j) - \gamma'_{\theta_0}(\lambda_j) b_T(j),\ j = 1, ..., \bar{T},$$

are the forward least squares residuals and

$$b_T(j) := A^{-1}_{\theta_0,T}(j) \frac{1}{\tilde{T}} \sum_{k=j+1}^{\tilde{T}} \gamma_{\theta_0}(\lambda_k) I_\varepsilon(\lambda_k).$$

It is worth observing that the motivation to employ only the first $\bar{T}$ Fourier frequencies to compute the recursive residuals is due to the singularity of $A_{\theta,T}(j)$ for all $j > \bar{T}$.

The empirical process β_T^0 can be written as a linear transformation of α_T^0,

$$\beta_T^0(\lambda) = \mathcal{L}_{\theta_0,T}\alpha_T^0(\lambda), \; \lambda \in [0,\pi],$$

where, for any function $g \in D[0,\pi]$,

$$\mathcal{L}_{\theta,T}g(\lambda) = g\left(\frac{\bar{T}}{\tilde{T}}\lambda\right) - \frac{1}{\tilde{T}}\sum_{j=1}^{[\bar{T}\lambda/\pi]}\gamma_\theta'(\lambda_j)A_{\theta,T}^{-1}(j)\int_{\lambda_{j+1}}^{\pi}\gamma_\theta\left(\tilde{\lambda}\right)g\left(d\tilde{\lambda}\right).$$

The transformation $\mathcal{L}_{\theta_0,T}$ has the limiting version $\mathcal{L}^0$, defined as

$$\mathcal{L}^0 g(\lambda) = g(\lambda) - \frac{1}{\pi}\int_0^\lambda \gamma_{\theta_0}'\left(\bar{\lambda}\right)A_{\theta_0}^{-1}\left(\bar{\lambda}\right)\int_{\bar{\lambda}}^{\pi}\gamma_{\theta_0}\left(\tilde{\lambda}\right)g\left(d\tilde{\lambda}\right)d\bar{\lambda},$$

where

$$A_{\theta_0}(\lambda) := \int_\lambda^\pi \gamma_{\theta_0}\left(\tilde{\lambda}\right)\gamma_{\theta_0}'\left(\tilde{\lambda}\right)d\tilde{\lambda}.$$

Notice that $\mathcal{L}^0\alpha_\infty$ is the martingale innovation of α_∞, see Khmaladze (1981).

This type of martingale transformation has been used by Khmaladze (1981) and Aki (1986) in the standard goodness-of-fit testing problem, by Nikabadze and Stute (1997) for goodness-of-fit of distribution functions under random censorship, by Stute, Thies and Zhu (1998), Koul and Stute (1998, 1999) and Khmaladze and Koul (2004) for dynamic regression models, and by Stute and Zhu (2002) for generalized linear models.

Henceforth, $B_\pi(\lambda) := B(\lambda/\pi)$ for $\lambda \in [0,\pi]$.

Theorem 3 *Under H_0 and assuming $A1 - A5$,*

$$\beta_T^0 \Rightarrow B_\pi.$$

Because β_T^0 cannot be computed in practice, as it depends on θ_0, it is suggested to use $\beta_{\theta_T,T}$, where

$$\begin{aligned}\beta_{\theta,T}(\lambda) &:= \mathcal{L}_{\theta,T}\alpha_{\theta,T}(\lambda) \\ &= \frac{2\pi}{G_{\theta,T}(\pi)}\frac{1}{\tilde{T}^{1/2}}\sum_{j=1}^{[\bar{T}\lambda/\pi]} e_{\theta,T}(j), \; \lambda \in [0,\pi]\end{aligned}$$

and

$$e_{\theta,T}(j) = \frac{I_X(\lambda_j)}{h_\theta(\lambda_j)} - \gamma_\theta'(\lambda_j)b_{\theta,T}(j), \; j = 1, ..., \bar{T},$$

are the forward recursive residuals in the linear projection of $I_X(\lambda_j)/h_\theta(\lambda_j)$ on $\gamma_\theta(\lambda_j)$, and where

$$b_{\theta,T}(j) = A_{\theta,T}^{-1}(j)\frac{1}{\widetilde{T}}\sum_{k=j+1}^{\tilde{T}}\gamma_\theta(\lambda_k)\frac{I_X(\lambda_k)}{h_\theta(\lambda_k)}.$$

In order to establish the asymptotic equivalence between β_T^0 and $\beta_{\theta_T,T}$, we also need some extra smoothness assumptions on the model under the null.

A6 For some $0<\delta<1$ and all $\lambda\in(0,\pi]$, there exists a constant $K<\infty$ such that

$$\sup_{\{\theta:\|\theta-\theta_0\|\le\delta\}}\frac{1}{\|\theta-\theta_0\|^2}\left\|\phi_\theta(\lambda)-\phi_{\theta_0}(\lambda)-\dot{\phi}_{\theta_0}(\lambda)(\theta-\theta_0)\right\|\le K|\log\lambda|,$$

and $\dot{\phi}_\theta$ satisfies $A3(a)-(c)$.

This assumption holds for all models used in practice, like ARFIMA in (3), Bloomfield's exponential and the fractional noise models mentioned before. In fact, they satisfy even the stronger condition with $K|\log\lambda|$ replaced by K.

Theorem 4 *Under H_0 and assuming $A1-A6$,*

$$\sup_{\lambda\in[0,\pi]}\left|\beta_{\theta_T,T}(\lambda)-\beta_T^0(\lambda)\right| = o_p(1).$$

Theorem 4 holds true, mutatis mutandis, with θ_T replaced by any $T^{1/2}$-consistent estimator. Also, from a computational point of view, it is worth observing that

$$A_{\theta,T}^{-1}(j) = A_{\theta,T}^{-1}(j+1) - \frac{A_{\theta,T}^{-1}(j+1)\gamma_\theta(\lambda_j)\gamma_\theta'(\lambda_j)A_{\theta,T}^{-1}(j+1)}{\widetilde{T}+\gamma_\theta'(\lambda_j)A_{\theta,T}^{-1}(j+1)\gamma_\theta(\lambda_j)},$$

and

$$b_{\theta,T}(j) = b_{\theta,T}(j+1) + A_{\theta,T}^{-1}(j)\gamma_\theta(\lambda_j)\left[\frac{I_X(\lambda_j)}{h_\theta(\lambda_j)}-\gamma_\theta'(\lambda_j)b_{\theta,T}(j+1)\right],$$

see Brown, Durbin and Evans (1975) for similar arguments.

Alternatively to $\beta_{\theta_T,T}$, we could have considered the cusum of backward recursive residuals, i.e.

$$\bar{\beta}_{\theta_T,T}(\lambda) := \frac{2\pi}{G_{\theta_T,T}(\pi)}\frac{1}{\widetilde{T}^{1/2}}\sum_{j=p+1}^{[\widetilde{T}\lambda/\pi]}\bar{e}_{\theta_T,T}(j),\ \lambda\in[0,\pi],$$

where

$$\bar{e}_{\theta,T}(j) := \frac{I_X(\lambda_j)}{h_\theta(\lambda_j)} - \gamma_\theta'(\lambda_j)\bar{b}_{\theta,T}(j),\ j=p+1,...,\widetilde{T},$$

$$\bar{b}_{\theta,T}(j) := \bar{A}^{-1}_{\theta,T}(j)\frac{1}{\tilde{\bar{T}}}\sum_{k=1}^{j-1}\gamma_\theta(\lambda_k)\frac{I_X(\lambda_k)}{h_\theta(\lambda_k)} \text{ and } \bar{A}_{\theta,T}(j) := \frac{1}{\tilde{\bar{T}}}\sum_{k=1}^{j-1}\gamma_\theta(\lambda_k)\gamma'_\theta(\lambda_k).$$

In this case, we can take advantage of the computational formulae,

$$\bar{A}^{-1}_{\theta,T}(j+1) = \bar{A}^{-1}_{\theta,T}(j) - \frac{\bar{A}^{-1}_{\theta,T}(j)\gamma_\theta(\lambda_{j+1})\gamma'_\theta(\lambda_{j+1})\bar{A}^{-1}_{\theta,T}(j)}{\tilde{\bar{T}} + \gamma'_\theta(\lambda_{j+1})\bar{A}^{-1}_{\theta,T}(j)\gamma_\theta(\lambda_{j+1})}$$

and

$$\bar{b}_{\theta,T}(j+1) = \bar{b}_{\theta,T}(j) + \bar{A}^{-1}_{\theta,T}(j+1)\gamma_\theta(\lambda_{j+1})\left[\frac{I_X(\lambda_{j+1})}{h_\theta(\lambda_{j+1})} - \gamma'_\theta(\lambda_{j+1})\bar{b}_{\theta,T}(j)\right].$$

This formulation may be useful in small samples when we suspect that the main discrepancy between the null and the alternative is near π. However, from Theorems 3 and 4, it is easily seen that the empirical processes $\bar{\beta}_{\theta_T,T}$ and $\beta_{\theta_T,T}$ have the same asymptotic behavior.

Let $\varphi : D[0,\pi] \to \mathbb{R}$ be a continuous functional, under H_0 and the conditions in Theorem 4,

$$\varphi\left(\beta_{\theta_T,T}\right) \xrightarrow{d} \varphi(B_\pi),$$

as a consequence of the continuous mapping theorem. For instance,

$$\begin{aligned}
\hat{K}_T &= \sup_{j=1,\dots,\bar{T}}\left|\beta_{\theta_T,T}\left(\frac{j\pi}{\bar{T}}\right)\right| \xrightarrow{d} \sup_{\lambda\in[0,\pi]}|B_\pi(\lambda)| \stackrel{d}{=} \sup_{\omega\in[0,1]}|B(\omega)|,\\
\hat{C}_T &= \frac{1}{\bar{T}}\sum_{j=1}^{\bar{T}}\beta_{\theta_T,T}\left(\frac{j\pi}{\bar{T}}\right)^2 \xrightarrow{d} \frac{1}{\pi}\int_0^\pi B_\pi^2(\lambda)\,d\lambda \stackrel{d}{=} \int_0^1 B^2(\omega)\,d\omega.
\end{aligned}$$

The above limiting distributions are tabulated, see e.g. Shorack and Wellner (1986, pp. 34 and 748.)

3. LOCAL ALTERNATIVES: OMNIBUS, DIRECTIONAL AND PORTMANTEAU TESTS

In this section, we shall show that tests based on $\beta_{\theta_T,T}$ are able to detect local alternatives of the type

$$H_{1T}:\ h(\lambda) = h_{\theta_0}(\lambda)\left(1 + \tau\frac{1}{\tilde{T}^{1/2}}l(\lambda) + \frac{1}{\tilde{T}}s_T(\lambda)\right),\ \lambda\in[0,\pi] \text{ and for some } \theta_0\in\Theta,$$

where $\int_0^\pi l(\lambda)\,d\lambda = 0$, $l(\lambda)$ satisfies the same properties as ϕ_{θ_0} in $A3(a)-(c)$, τ is a constant, possibly unknown, and for some finite T_0, $\sup_{T>T_0}|s_T(\cdot)|$ is an integrable function. Let us consider some examples.

Example 1 *If we wish to study departures of the white noise hypothesis in the direction of fractional alternatives, we have that*

$$\frac{h(\lambda)}{h_{\theta_0}(\lambda)} = \frac{1}{|2\sin(\lambda/2)|^{2d/\widetilde{T}^{1/2}}},\ \lambda \in [0,\pi],$$

for some $d \neq 0$*. By a simple Taylor's expansion up to its second term,*

$$l(\lambda) = -2\log|2\sin(\lambda/2)| \ \text{ and } \tau = d,$$

respectively, with the remainder function s_T *being such that for some* $0 \le \epsilon < 1$, $|s_T(\lambda)| \le K|\lambda|^{-\epsilon}$ *for all large* T *and some* $K < \infty$.

Example 2 *If we consider departures in the direction of* $MA(1)$ *alternatives, we obtain that*

$$\frac{h(\lambda)}{h_{\theta_0}(\lambda)} = 1 - \eta\frac{1}{\widetilde{T}^{1/2}}2\cos(\lambda) + \frac{1}{\widetilde{T}}\eta^2,\ \lambda \in [0,\pi].$$

Thus, $\tau = \eta$, $l(\lambda) = -2\cos(\lambda)$ *and* $s_T(\lambda) = \eta^2$.

Example 3 *If we consider departures in the direction of* $AR(1)$ *alternatives, then*

$$\frac{h(\lambda)}{h_{\theta_0}(\lambda)} = \left[1 - \delta\frac{1}{\widetilde{T}^{1/2}}2\cos(\lambda) + \frac{1}{\widetilde{T}}\delta^2\right]^{-1},\ \lambda \in [0,\pi].$$

Thus, $\tau = \delta$ *and* $l(\lambda) = 2\cos(\lambda)$ *with* $|s_T(\lambda)| \le K$*, for all large* T *and some* $K < \infty$.

For $\lambda \in [0,\pi]$, let us define

$$L(\lambda) := \frac{1}{\pi}\int_0^\lambda \left\{ l(\bar{\lambda}) - \gamma'_{\theta_0}(\bar{\lambda})\, A^{-1}_{\theta_0}(\bar{\lambda})\, \frac{1}{\pi}\int_{\bar{\lambda}}^{\pi} \gamma_{\theta_0}\left(\tilde{\lambda}\right) l\left(\tilde{\lambda}\right) d\tilde{\lambda} \right\} d\bar{\lambda} \tag{10}$$

and

$$M(\lambda) := B_\pi(\lambda) + \tau \cdot L(\lambda),\ \lambda \in [0,\pi].$$

We have the following theorem.

Theorem 5 *Assuming the same assumptions as in Theorem 4, under* H_{1T},

$$\beta_{\theta_T,T} \Rightarrow M.$$

Using the fact that M and B_π are identically distributed, except for the deterministic shift $\tau \cdot L$, and taking into account that $2^{1/2}\sin((j-1/2)\lambda)$ and $1/(j-1/2)^2\pi^2$ are the eigenfunctions and eigenvalues in the Kac-Siegert representation of B_π (Kac and Siegert, 1947), the orthogonal components of M

$$m(j) := 2^{1/2}\left(j-\frac{1}{2}\right)\int_0^\pi \sin\left(\left(j-\frac{1}{2}\right)\lambda\right) M(\lambda)\,d\lambda,\ j=1,2,...$$

are independently distributed normal random variables with mean $\tau \cdot \ell(j)$ and variance 1, where

$$\ell(j) = 2^{1/2}\left(j-\frac{1}{2}\right)\int_0^\pi \sin\left(\left(j-\frac{1}{2}\right)\lambda\right) L(\lambda)\,d\lambda,\ j=1,2,....$$

Using, the (asymptotically) orthogonal components of $\beta_{\theta_T,T}$,

$$\tilde{m}_T(j) = 2^{1/2}\left(j-\frac{1}{2}\right)\int_0^\pi \sin\left(\left(j-\frac{1}{2}\right)\lambda\right) \beta_{\theta_T,T}(\lambda)\,d\lambda,\ j=1,2,...,$$

we obtain the spectral representation,

$$\beta_{\theta_T,T}(\lambda) = 2^{1/2}\sum_{j=1}^{\infty}\frac{\tilde{m}_T(j)\sin\left(\left(j-\frac{1}{2}\right)\lambda\right)}{\pi\left(j-\frac{1}{2}\right)},\ \lambda\in[0,\pi].$$

By Theorem 5 and the continuous mapping theorem, finitely many of the $\tilde{m}_T(j)$´s converge in distribution to the corresponding $m(j)$´s under H_{1T}. Using Parseval's Theorem,

$$\hat{C}_T \xrightarrow{d} \sum_{j=1}^{\infty}\frac{m^2(j)}{(j-\frac{1}{2})^2\pi^2}.$$

Using similar arguments to those in Eubank and LaRicca (1992) in the context of the standard empirical process with estimated parameters, tests based on

$$\tilde{W}_{n,T} := \sum_{j=1}^{n}\tilde{m}_T^2(j),$$

with a reasonable choice of $n \geq 1$, will lead to gains in power, compared to $\hat{C}_T$, in the direction of alternatives with significant autocorrelations at high lags. These Portmanteau tests are related to Neyman's (1937) smooth tests, a compromise between omnibus and directional tests, and for each $n \geq 1$, under H_{1T}, we have that

$$\tilde{W}_{n,T} \xrightarrow{d} \chi_n^2\left(\tau^2\sum_{j=1}^{n}\ell^2(j)\right).$$

That is, tests based on $\tilde{W}_{n,T}$ are asymptotically pivotal under H_0 ($\tau = 0$) for each choice of n, and more importantly, they are able to detect local alternatives converging to the null at the parametric rate $T^{-1/2}$, provided that $\ell(j) \neq 0$ for some $j = 1, ..., n$. The latter is in contrast with the classical Portmanteau tests based on

$$\tilde{Q}_{n_T,T} := \sum_{j=1}^{n_T} \left(T^{1/2} \tilde{\rho}_T(j) \right)^2, \tag{11}$$

where $\tilde{\rho}_T(j)$ is some estimate of the $j - th$ autocorrelation of the residuals. It has been shown that $\tilde{Q}_{n_T,T}$ is approximately distributed as a $\chi^2_{n_T-p}$ under H_0 specifying a short-range model and assuming that n_T diverges as $T \to \infty$. On the other hand, the resulting test is able to detect alternatives converging to the null at the rate $n_T^{1/4} T^{-1/2}$ (see e.g. Hong 1986), which is slower than $T^{-1/2}$.

In practice, it is recommendable to use the discrete version

$$\hat{W}_{n,T} := \sum_{j=1}^{n} \hat{m}_T^2(j)$$

of $\tilde{W}_{n,T}$, with

$$\hat{m}_T(j) := 2^{1/2} \left(j - \frac{1}{2} \right) \cdot \frac{\pi}{\bar{T}} \sum_{k=1}^{\bar{T}} \sin \left(\left(j - \frac{1}{2} \right) \frac{\pi k}{\bar{T}} \right) \beta_{\theta_T,T} \left(\frac{\pi k}{\bar{T}} \right).$$

On the other hand, optimal tests of H_0 in the direction H_{1T} can be constructed applying results in Grenander (1950) (see also Grenander 1981, and references therein), as was suggested by Stute (1997) in the context of goodness-of-fit testing of a regression function. Asymptotically, testing for H_0 in the direction of H_{1T} is equivalent to test $\bar{H}_0 : \mathbb{E}(m(j)) = 0$ for all $j \geq 1$, against $\bar{H}_1 : \mathbb{E}(m(j)) = \tau \cdot \ell(j)$ for all $j \geq 1$ with L known, but maybe with unknown τ. Under $\bar{H}_0$, the distribution of $\{m(j)\}_{j \geq 1}$ is completely specified, as is also under $\bar{H}_1$ when the parameter τ is known. Then, the likelihood-ratio for a finite dimensional set $(m(1), ..., m(n))$ is

$$\Lambda_n = \exp \left(\tau \sum_{j=1}^{n} \ell(j) \cdot \left(m(j) - \frac{\tau \cdot \ell(j)}{2} \right) \right). \tag{12}$$

Grenander (1950) showed that $\Lambda_n \to_p \Lambda_\infty$ as $n \to \infty$, and that the most powerful test at the α significance level has a critical region of the form $\{\Lambda_\infty > k\}$, with $P_0 \{\Lambda_\infty > k\} = \alpha$ if $\sum_{j=1}^{\infty} \ell^2(j) < \infty$. The latter condition is satisfied in our context by Parseval's Theorem and A3(c) because l is a square integrable function.

Define

$$\psi := \frac{\sum_{j=1}^{\infty} \ell(j) \cdot m(j)}{\left(\sum_{j=1}^{\infty} \ell^2(j) \right)^{1/2}}.$$

Then under H_0, $\psi \overset{d}{=} N(0,1)$, and in view of (12), ψ forms a basis to obtain optimal critical regions. When the sign of τ is known, the critical region of the uniformly most powerful test at the α significance level is $\{\psi > z_{1-\alpha}\}$ when $\tau > 0$ and $\{\psi < -z_{1-\alpha}\}$ when $\tau < 0$, where z_{υ} is the υ quantile of the standard normal. Also, when the sign of τ is unknown, the most powerful unbiased test at the α significance level has critical region given by $\left\{|\psi| > z_{1-\alpha/2}\right\}$.

These arguments suggest an (asymptotically) optimal Neyman-Pearson test in the direction of H_{1T} based on the first n orthogonal components of $\beta_{\theta_T,T}$, using the test statistic

$$\hat{\psi}_{n,T} = \frac{\sum_{j=1}^{n} \ell(j) \cdot \hat{m}_T(j)}{\left(\sum_{j=1}^{n} \ell^2(j)\right)^{1/2}}.$$

Schoenfeld (1977) proposes the same type of statistic in the standard goodness-of-fit testing context. Under H_0 and the assumptions in previous sections, we have that

$$\hat{\psi}_{n,T} \xrightarrow{d} N(0,1) \text{ as } T \to \infty \text{ for each fixed } n.$$

Also, arguing as in Schoenfeld's (1977) Theorem 3, it can be shown the convergence in distribution of $\hat{\psi}_{n_T,T}$ when n_T increases with T. Approximately optimal tests for H_0 in the direction of H_{1T} reject H_0 at the α significance level when $\left|\hat{\psi}_{n_T,T}\right| > z_{1-\alpha/2}$ if τ has unknown sign, $\hat{\psi}_{n_T,T} > z_{1-\alpha}$ when $\tau > 0$ and $\hat{\psi}_{n_T,T} < -z_{1-\alpha}$ when $\tau < 0$.

4. SOME MONTE CARLO EXPERIMENTS

A small Monte-Carlo study has been carried out to investigate the finite sample performance of the different tests. To that end, we have considered the AR(1), MA(1) and ARFIMA$(0, d_0, 0)$ models

$$(1 - \delta_0 L) X(t) = \varepsilon(t), \tag{13}$$

$$X(t) = (1 - \eta_0 L) \varepsilon(t), \tag{14}$$

$$(1 - L)^{d_0} X(t) = \varepsilon(t), \tag{15}$$

respectively, where the parameter θ_0 equals to δ_0, η_0 and d_0 for the different models and L is the lag operator. The innovations $\{\varepsilon(t)\}_{t=1}^{T}$ are *iid* $\mathcal{N}(0,1)$, and the sample sizes used are $T = 200$ and 500 with different values of the parameters δ_0, η_0 and d_0. For models (13) and (14), we have considered $\delta_0, \eta_0 = -0.8, -0.5, 0.0, 0.5, 0.8$, whereas for model (15), $d_0 = 0.0, 0.2, 0.4$. The ARFIMA model was simulated using an algorithm by Hosking (1984).

For the three models and all values of θ_0, we have computed the proportion of rejections in 50,000 generated samples for both sample sizes. Whittle estimates are obtained according to (7). For each of the models considered, ϕ_θ is given by

$$AR(1)\,,\ \theta=\delta:\phi_\delta(\lambda)=\frac{\partial}{\partial\delta}\log\left|1-\delta e^{i\lambda}\right|^{-2}=-2\frac{\delta-\cos\lambda}{1-2\delta\cos\lambda+\delta^2};$$
$$MA(1)\,,\ \theta=\eta:\phi_\eta(\lambda)=\frac{\partial}{\partial\eta}\log\left|1-\eta e^{i\lambda}\right|^{2}=2\frac{\eta-\cos\lambda}{1-2\eta\cos\lambda+\eta^2};$$
$$ARFIMA(0,d,0)\,,\ \theta=d:\phi_d(\lambda)=\frac{\partial}{\partial d}\log\left|1-e^{i\lambda}\right|^{-2d}=-2\log\left|2\sin\left(\lambda/2\right)\right|.$$

We also report, as a benchmark, the proportion of rejections using

$$C_T^0:=\frac{1}{\pi}\int_0^\pi\alpha^2_{\theta_0,T}\left(\lambda\right)d\lambda=T\sum_{j=1}^\infty\frac{\rho^2_{\theta_0,T}\left(j\right)}{\pi^2j^2},$$

which is suitable for testing simple hypotheses. In addition, for the sake of comparison, we provide the results for the Box and Pierce (1970) test statistic (11) using several values of n_T increasing with T, where $\tilde{\rho}_T\left(j\right)$, $j\geq1$, are the sample autocorrelations of the residuals $\left\{\hat{\varepsilon}\left(t\right)\right\}_{t=1}^T$. Specifically, for the AR(1) model,

$$\hat{\varepsilon}\left(t\right)=\left(1-\delta_TL\right)X\left(t\right),$$

with $X\left(t\right)=0$ for $t\leq0$; for the MA(1) model,

$$\hat{\varepsilon}\left(t\right)=X\left(t\right)-\eta_T\hat{\varepsilon}\left(t-1\right),$$

with $\hat{\varepsilon}\left(0\right)=0$, whereas for the ARFIMA$(0,d,0)$ model,

$$\hat{\varepsilon}\left(t\right)=\sum_{j=0}^{t-1}\vartheta(j,d_T)X(t-j),$$

where $\vartheta(j,d)$ are the coefficients in the formal expansion $(1-L)^d=\sum_{j=0}^\infty\vartheta(j,d)L^j$, with

$$\vartheta(j,d)=\frac{\Gamma\left(j-d\right)}{\Gamma\left(-d\right)\Gamma\left(j+1\right)};\quad\Gamma\left(a\right)=\int_0^\infty x^{a-1}e^{-x}dx.$$

The standardized values of $\tilde{Q}_{n_T,T}$, $\left(\tilde{Q}_{n_T,T}-n_T\right)/\sqrt{2n_T}$, are compared with the 5% critical value of the standard normal, see Hong (1996), instead of the usual $\chi^2_{(n_T-1)}$ approximation correcting by the loss of degrees of freedom due to parameter estimation, which is justified under Gaussianity. Both approximations provide similar proportion of rejections. We have also tried the weighting suggested by Ljung and Box (1978), which produced very similar results.

First we analyze the size accuracy of the Cramér-von Mises test based on $\beta_{\theta_T,T}$. The empirical sizes of the tests based on $\hat{C}_T$, reported in Table 1, are reasonably close to the nominal ones. The asymptotic approximation improves noticeably when the sample size increases from $T = 200$ to $T = 500$, being this improvement uniform for all the models, although the empirical size is smaller than the nominal level. Tests based on $\tilde{Q}_{n_T,T}$ have serious size distortions for the smaller sample size and large values of $|\eta|$ in the MA(1) model, since Whittle estimates can be quite biased in these cases. The empirical size of tests based on $\tilde{Q}_{n_T,T}$ depends substantially on the number of autocorrelations used. In addition, for the larger choices of n_T implemented, $\tilde{Q}_{n_T,T}$ over-rejects H_0. The usual recommendation $n_T = o\left(T^{1/2}\right)$ seems also reasonable here, in terms of size accuracy.

Next, we study the power performance of the tests. To this end, we report first, in Table 2, the proportion of rejections under the alternative hypothesis for different non-nested specifications with the model specified under the null. We cannot conclude that one test is clearly superior to the others in any of the four cases analyzed. As expected, the power of the Portmanteau test decreases as n_T increases. In view of Tables 1 and 2, we can conclude that a choice of large n_T, around $T^{-1/2}$, produces reasonable size accuracy, but such a choice is not the best possible one in order to maximize the power. The test based on $\hat{C}_T$ is fairly powerful compared to the Portmanteau test for all cases considered, and it works remarkably well when testing an AR(1) in the direction of a MA(1) alternative.

Finally, we analyze the power of the different tests when testing an AR(1) specification in the direction of local ARFIMA$(1, d, 0)$ with $d = \tau/T^{1/2}$, and in the direction of local ARMA$(1, 1)$ alternatives with moving average parameter $\eta = \tau/T^{1/2}$, for different values of τ. The proportion of rejections for these designs is reported in Tables 3 and 4. We also consider tests based on the test statistics $\hat{W}_{n,T}$ and $\hat{\psi}_{n,T}$ (one sided and two sided, $\hat{\psi}^{+}_{n,T}$ and $\left|\hat{\psi}_{n,T}\right|$ respectively) choosing $n = 3$ and 6, which has been recommended by Stute, Thies and Zhu (1998) for a different goodness-of-fit test problem. Of course, tests based on the first n (asymptotic) orthogonal components of $\beta_{\theta_T,T}$ are sensitive to the choice of n, as it also happens with tests based on the n (asymptotic) orthogonal components of $\alpha_{\theta_T,T}$ (the estimated autocorrelations of the innovations) in Portmanteau tests. The omnibus test based on $\hat{C}_T$ still works fairly well compared to the others, including the optimal and smooth tests. The directional tests are the most powerful in the directions for which they are designed, and the tests based on $\hat{W}_{n,T}$ and $\tilde{Q}_{n_T,T}$ work very similarly, though $\hat{W}_{n,T}$ exhibits a better size precision for the choices of n considered.

5. FINAL REMARKS

Our results can be extended to goodness-of-fit tests of models that can accommodate simultaneously stationary and non-stationary time series. For instance, if the increments $Y(t) := (1-L)X(t)$, $t = 0, \pm 1, \ldots$, are second order stationary with zero mean and spectral density g such that

$$\lim_{\lambda \to 0+} |\lambda|^{2(d-1)} g(\lambda) = G > 0 \text{ for some } d \in [0.5, 1.5),$$

we can define the pseudo-spectral density function of $\{X(t)\}_{t \in \mathbb{Z}}$, f, as

$$f(\lambda) = \frac{1}{|1 - e^{i\lambda}|^2} g(\lambda).$$

Thus, when $d \neq 1$, g has a singularity at $\lambda = 0$, as it happens with many long-range dependent time series (cf. A2). If $\{X(t)\}_{t \in \mathbb{Z}}$ is stationary, f becomes the standard spectral density function.

If either $\{Y(t)\}_{t \in \mathbb{Z}}$ or $\{X(t)\}_{t \in \mathbb{Z}}$ satisfy a Wold's decomposition, f admits the factorization

$$f(\lambda) = \frac{\sigma^2}{2\pi} h(\lambda),$$

where h satisfies $A2$. Thus, given a parametric family $\mathcal{H}$, for example the ARFIMA specification given in (3), a $T_p - process$ for testing that $h \in \mathcal{H}$ is

$$\alpha^w_{\theta_T,T}(\lambda) := \widetilde{T}^{1/2} \left[\frac{G^w_{\theta_T,T}(\lambda)}{G^w_{\theta_T,T}(\pi)} - \frac{\lambda}{\pi} \right], \ \lambda \in [0, \pi],$$

where $G^w_{\theta,T}$ is analogous to $G_{\theta,T}$, but using the tapered periodogram, e.g.

$$I^w_X(\lambda) := \frac{\left| \sum_{t=1}^T w(t) X(t) e^{it\lambda} \right|^2}{2\pi \sum_{t=1}^T w^2(t)}.$$

Here $\theta_T = \arg\min_{\theta \in \Theta} G^w_{\theta,T}(\pi)$ is the Whittle estimator proposed by Velasco and Robinson (2000), which admits a similar asymptotic first order expansion as in (8), and where w is a taper function, e.g. the full cosine taper

$$w(t) = \frac{1}{2}\left(1 - \cos\left(\frac{2\pi t}{T}\right)\right), \quad t = 1, \ldots, T.$$

If the full cosine taper is used, because of its desirable asymptotic properties (see Velasco, 1999), it is recommended in practice to base our tests on the empirical process $\beta^{w}_{\theta_T,T}$, where

$$\beta^{w}_{\theta,T}(\lambda_m) := \left(\frac{P_4^2}{\tilde{T}}\right)^{1/2} \frac{2\pi}{G^{w}_{\theta,T}(\pi)} \sum_{j=1}^{m} e^{w}_{\theta,T}(j), \ m = 1, ..., \overline{T},$$

with

$$e^{w}_{\theta,T}(j) := \frac{I^{w}_{X}(\lambda_j)}{h_\theta(\lambda_j)} - \gamma'_\theta(\lambda_j)\, b^{w}_{\theta,T}(j), \ b^{w}_{\theta,T}(j) := A^{-1}_{\theta,T}(j)\, \frac{1}{\widetilde{\widetilde{T}}} \sum_{k=j+1}^{\widetilde{T}} \gamma_\theta(\lambda_k)\, \frac{I^{w}_{X}(\lambda_k)}{h_\theta(\lambda_k)},$$

and

$$P_4^2 := \lim_{T\to\infty} \frac{T \sum_{t=1}^{T} w^4(t)}{\left(\sum_{t=1}^{T} w^2(t)\right)^2} = \frac{35}{18}.$$

Under appropriate regularity conditions, it can be proved using tools in Velasco (1999) and Velasco and Robinson (2000) that $\beta^{w}_{\theta_T,T} \Rightarrow B_\pi$.

Finally, the methodology can be extended to test the correlation structure of the innovations of regression models (e.g. distributed-lags models) using the martingale part of the $U_p - process$ based on the residuals. When $\mathbb{E}(z(t)\, u(s)) = 0$ for all t, s, where $\{z(t)\}_{t=1}^{T}$ are the regressors and $\{u(t)\}_{t=1}^{T}$ the error term, the residual $U_p - process$ is asymptotically equivalent to the $U_p - process$ based on the true innovations, and there is no need of using tests based on the martingale part of the $U_p - process$. When $\mathbb{E}(z(t)\, u(t-s)) \neq 0$ for some $s > 0$, the first order expansion of the residuals $U_p - process$ depends on the cross-spectrum of the innovations and regressors. However, it seems possible to apply the results in this paper to implement tests based on the (approximate) martingale part of this $U_p - process$ with estimated parameters.

6. LEMMAS

This section provides a series of lemmas which will be used in the proofs of the main results. Some of them can be of independent interest. Henceforth, $z^{(k)}$ denotes the $k - th$ element of a $p \times 1$ vector z and K a finite positive constant. Also, we shall abbreviate $g(\lambda_j)$ by g_j for a generic function $g(\lambda)$.

Lemma 1 *Let* $\zeta : (0, \pi] \to \mathbb{R}^p$ *be a function such that* $\|\zeta(\lambda)\| \leq K\, |\log \lambda|^{\ell}$, $\ell \geq 1$, *and*

$\|\partial\zeta(\lambda)/\partial\lambda\| \leq K\lambda^{-1}|\log\lambda|^{\ell-1}$ *for all* $\lambda > 0$*. Then, as* $T \to \infty$*,*

$$\sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\widetilde{T}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\zeta_j - \frac{1}{\pi}\int_0^\lambda \zeta(x)\,dx\right\| \leq K\frac{\left(\log\widetilde{T}\right)^\ell}{\widetilde{T}}. \tag{16}$$

Proof. The left side of (16) is bounded by

$$\sup_{\lambda\in[0,\pi/\widetilde{T})}\left\|\frac{1}{\pi}\int_0^\lambda \zeta(x)\,dx\right\| + \sup_{\lambda\in[\pi/\widetilde{T},\pi]}\left\|\frac{1}{\widetilde{T}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\zeta_j - \frac{1}{\pi}\int_0^\lambda \zeta(x)\,dx\right\|. \tag{17}$$

The first term of (17) is bounded by

$$\frac{1}{\pi}\int_0^{\pi/\widetilde{T}}\|\zeta(x)\|\,dx \leq K\int_0^{\pi/\widetilde{T}}|\log x|^\ell\,dx \leq K\frac{\left(\log\widetilde{T}\right)^\ell}{\widetilde{T}}.$$

Next, by the triangle inequality, the second term of (17) is bounded by

$$\sup_{\lambda\in[\pi/\widetilde{T},\pi]}\left\|\frac{1}{\widetilde{T}}\zeta(\lambda) - \frac{1}{\pi}\int_0^{\pi/\widetilde{T}}\zeta(x)\,dx\right\| + \sup_{\lambda\in[\pi/\widetilde{T},\pi]}\frac{1}{\pi}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]-1}\int_{j\pi/\widetilde{T}}^{(j+1)\pi/\widetilde{T}}\left\|\zeta_j - \zeta(x)\right\|dx. \tag{18}$$

The first term of (18) is bounded by $K\widetilde{T}^{-1}\left(\log\widetilde{T}\right)^\ell$ since $\|\zeta(x)\| \leq K|\log x|^\ell$. Next, by the mean value theorem, the second term of (18) is bounded by

$$\begin{aligned}
K\sum_{j=1}^{\widetilde{T}-1}\int_{j\pi/\widetilde{T}}^{(j+1)\pi/\widetilde{T}}\frac{1}{\lambda_j}\left|\frac{j\pi}{\widetilde{T}} - x\right||\log x|^{\ell-1}\,dx &\leq K\sum_{j=1}^{\widetilde{T}-1}\frac{1}{j}\int_{j\pi/\widetilde{T}}^{(j+1)\pi/\widetilde{T}}|\log x|^{\ell-1}\,dx \\
&\leq \frac{K\left(\log\widetilde{T}\right)^\ell}{\widetilde{T}}.
\end{aligned}$$

□

The next lemma corresponds to Giraitis, Hidalgo and Robinson's (2001) Lemma 4.4, which we state, without proof, for easy reference. For this purpose, let $u_j := h_j^{-1/2}(2\pi T)^{-1/2}\sum_{t=1}^T X(t)e^{it\lambda_j}$ $v_j := (2\pi T)^{-1/2}\sum_{t=1}^T \varepsilon(t)e^{it\lambda_j}$ and $R_{X\varepsilon}(\lambda)$ the spectral coherency (Brillinger, 1981, pp. 256-257) between X and ε. Also herewith $\overline{c}$ will denote the conjugate of the complex number c.

Lemma 2 *Assuming A1 and A2, then, as* $T \to \infty$*, the following relations hold uniformly* *over* $1 \leq j < k \leq \widetilde{T}$:

$$\begin{aligned}
\mathbb{E}\left(u_j \overline{v}_j\right) &= R_{X\varepsilon,j} + O\left(j^{-1}\log\left(j+1\right)\right); \\
\mathbb{E}\left(u_j v_j\right) &= O\left(j^{-1}\log\left(j+1\right)\right); \\
\max\left(\left|\mathbb{E}\left(u_k \overline{v}_j\right)\right|, \left|\mathbb{E}\left(u_k v_j\right)\right|\right) &= O\left(j^{-1}\log\left(k\right)\right); \\
\max\left(\left|\mathbb{E}\left(v_k \overline{u}_j\right)\right|, \left|\mathbb{E}\left(v_k u_j\right)\right|\right) &= O\left(j^{-1}\log\left(k\right)\right).
\end{aligned}$$

The next lemma corresponds to the proof of expression (4.8) of Robinson (1995b, pp. 1648-1651), using the orders of magnitude of the terms a_1, a_2, b_1 and b_2 in Robinson (1995b) and his Lemma 3, but using our Lemma 2 instead of Robinson's (1995*a*) Theorems 1 and 2 when appropriate.

Lemma 3 *Let* $\zeta : [0,\pi] \to \mathbb{R}^p$ *satisfy the same conditions of* ϕ_{θ_0} *in A3* $(a)-(c)$. *Then, assuming A1 and A2, as* $T \to \infty$, *for* $1 \leq r < s \leq \widetilde{T}$, $h = 1, \ldots, p$:

$$\mathbb{E}\left|\sum_{j=r}^{s} \zeta_j^{(h)} v_j \left(\overline{u}_j - \overline{v}_j\right)\right|^2 \leq K \log^2\left(T\right) \sum_{j=r}^{s} \left\{ j^{-1}\log\left(T\right) + \sum_{k=r}^{s} \left(j^{-2}\log^2\left(T\right) + j^{-1}k^{-1/2} \right) \right\}.$$

Lemma 4 *Let* $\zeta : [0,\pi] \to \mathbb{R}^p$ *satisfy the same conditions of* ϕ_{θ_0} *in A3* $(a)-(c)$ *and write*

$$\alpha_T^{\zeta}(\lambda) := \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j \left(I_{\varepsilon,j} - \frac{\sigma^2}{2\pi} \right), \quad \tilde{\alpha}_T^{\zeta}\left(\lambda\right) := \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j \left(\frac{I_{X,j}}{h_j} - \frac{\sigma^2}{2\pi} \right).$$

Then, under the conditions of Theorem 1, for some $0 < \delta < 1/6$,

$$\mathbb{E} \sup_{\lambda \in [0,\pi]} \left\| \tilde{\alpha}_T^{\zeta}\left(\lambda\right) - \alpha_T^{\zeta}\left(\lambda\right) \right\| = O\left(T^{-\delta}\right). \tag{19}$$

Proof. It suffices to show that (19) holds true for each element of the vector $\tilde{\alpha}_T^{\zeta}\left(\lambda\right) - \alpha_T^{\zeta}\left(\lambda\right)$. Then, by the triangle inequality, the left side of (19) is bounded by

$$\mathbb{E} \sup_{\lambda \in [0,\pi]} \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \left|\zeta_j^{(k)}\right| \left|u_j - v_j\right|^2 + 2\mathbb{E} \sup_{\lambda \in [0,\pi]} \left| \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right|. \tag{20}$$

The first term of (20) is bounded by

$$\begin{aligned}
&\frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{\widetilde{T}} \left|\zeta_j^{(k)}\right| \left\{ \left(\mathbb{E}\left|u_j\right|^2 - \frac{\sigma^2}{2\pi} \right) - \left(\mathbb{E}\left(u_j \overline{v}_j\right) - \frac{\sigma^2}{2\pi} \right) \right. \\
&\qquad \left. - \left(\mathbb{E}\left(\overline{u}_j v_j\right) - \frac{\sigma^2}{2\pi} \right) + \left(\mathbb{E}\left|v_j\right|^2 - \frac{\sigma^2}{2\pi} \right) \right\} \\
&= O\left(\frac{\log T}{\widetilde{T}^{1/2}} \sum_{j=1}^{\widetilde{T}} \frac{\log\left(j+1\right)}{j} \right) = O\left(T^{-\delta}\right),
\end{aligned}$$

by Lemma 2, because $\mathbb{E}\left|v_j\right|^2 = (2\pi)^{-1}\sigma^2$ and by assumption, $\left|\zeta_j^{(k)}\right| \leq K \log T$.

Next, to show that the second term of (20) is $O\left(T^{-\delta}\right)$, it suffices to show that

$$\mathbb{E} \max_{s=1,\ldots,\widetilde{T}} \left| \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{s} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right| = O\left(T^{-\delta}\right). \tag{21}$$

By the triangle inequality, the left side of (21) is bounded by

$$\mathbb{E} \max_{s=1,\ldots,\left[\widetilde{T}^{\beta}\right]} \left| \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{s} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right| + \mathbb{E} \left| \frac{1}{\widetilde{T}^{1/2}} \sum_{j=1}^{\left[\widetilde{T}^{\beta}\right]} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right| \tag{22}$$

$$+\mathbb{E} \max_{s=\left[\widetilde{T}^{\beta}\right]+1,\ldots,\widetilde{T}} \left| \frac{1}{\widetilde{T}^{1/2}} \sum_{j=\left[\widetilde{T}^{\beta}\right]+1}^{s} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right|, \tag{23}$$

where $\frac{1}{3} < \beta < \frac{1}{2}$. Using the inequality

$$\left(\sup_p \left|c_p\right|\right)^2 = \sup_p \left|c_p\right|^2 \leq \sum_p \left|c_p\right|^2, \tag{24}$$

by the Cauchy-Schwarz inequality, the square of (22) is bounded by

$$\frac{4}{\widetilde{\widetilde{T}}} \sum_{s=1}^{\left[\widetilde{T}^{\beta}\right]} \mathbb{E} \left| \sum_{j=1}^{s} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right|^2 = O\left(\widetilde{T}^{2\beta-1} \log^4 T\right) = O\left(T^{-2\delta}\right)$$

using Lemma 3.

To complete the proof, we need to show that (23) $= O\left(T^{-\delta}\right)$. To that end, let $q = 0, \ldots, \left[\widetilde{T}^{\varsigma}\right] - 1$ with $\frac{1}{3} < \varsigma < \beta$. By the triangle inequality, (23) is bounded by

$$\mathbb{E} \frac{1}{\widetilde{T}^{1/2}} \max_{s=\left[\widetilde{T}^{\beta}\right]+1,\ldots,\widetilde{T}} \left| \left\{ \sum_{j=\left[\widetilde{T}^{\beta}\right]+1}^{s} - \sum_{j=\left[\widetilde{T}^{\beta}\right]+1}^{\left[\widetilde{T}^{\beta}\right]+q(s)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]} \right\} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right| \tag{25}$$
$$+\mathbb{E} \frac{1}{\widetilde{T}^{1/2}} \max_{s=\left[\widetilde{T}^{\beta}\right]+1,\ldots,\widetilde{T}} \left| \sum_{j=\left[\widetilde{T}^{\beta}\right]+1}^{\left[\widetilde{T}^{\beta}\right]+q(s)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right|$$

where $q(s)$ denotes the value of $q = 0, \ldots, \left[\widetilde{T}^{\varsigma}\right] - 1$ such that $\left[\widetilde{T}^{\beta}\right] + q(s)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]$ is the largest integer smaller than or equal to s, and using the convention $\sum_c^d \equiv 0$ if $d < c$.

By the definition of $q\left(s\right)$ and the Cauchy-Schwarz inequality, the square of the second term of (25) is bounded by

$$\mathbb{E} \frac{1}{\widetilde{\widetilde{T}}} \max_{q=0,\ldots,\left[\widetilde{T}^{\varsigma}\right]-1} \left| \sum_{j=\left[\widetilde{T}^{\beta}\right]+1}^{\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right|^2 \leq \frac{1}{\widetilde{\widetilde{T}}} \sum_{q=0}^{\left[\widetilde{T}^{\varsigma}\right]-1} \mathbb{E} \left| \sum_{j=\left[\widetilde{T}^{\beta}\right]+1}^{\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]} \zeta_j^{(k)} v_j \left(\overline{u}_j - \overline{v}_j\right) \right|^2$$

by (24). But, using Lemma 3, we have that the right side of the last displayed inequality is bounded by

$$
\begin{aligned}
& K\frac{\log^4 T}{\widetilde{T}}\sum_{q=0}^{\left[\widetilde{T}^{\varsigma}\right]-1}\left(1+\frac{|q|_+\widetilde{T}^{1-\varsigma}}{\widetilde{T}^{\beta}}+|q|_+^{1/2}\widetilde{T}^{\frac{1}{2}(1-\varsigma)}\right) \\
\leq\ & K\log^4 T\left(\widetilde{T}^{\varsigma-\beta}+\widetilde{T}^{\varsigma-\frac{1}{2}}\right)\leq K\widetilde{T}^{-2\delta},
\end{aligned}
$$

where $|q|_+ = \max\{1,|q|\}$. To complete the proof we need to show that the first term in (25) is $O\left(T^{-\delta}\right)$. To that end, we note that this term is bounded by

$$
\mathbb{E}\frac{1}{\widetilde{T}^{1/2}}\max_{q=0,\ldots,\left[\widetilde{T}^{\varsigma}\right]-1}\max_{s}\left|\sum_{j=1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}^{s}\zeta_j^{(k)}v_j\left(\overline{u}_j-\overline{v}_j\right)\right|,
$$

where the $\max_s$ runs for all values $s = 1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right],\ldots,\left[\widetilde{T}^{\beta}\right]+(q+1)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]$. By the Cauchy-Schwarz inequality and (24), the square of the last displayed expression is bounded by

$$
\begin{aligned}
& \frac{1}{\widetilde{T}}\sum_{q=0}^{\left[\widetilde{T}^{\varsigma}\right]-1}\ \sum_{s=1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}^{\left[\widetilde{T}^{\beta}\right]+(q+1)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}\mathbb{E}\left|\sum_{j=1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}^{s}\zeta_j^{(k)}v_j\left(\overline{u}_j-\overline{v}_j\right)\right|^2 \\
\leq\ & K\frac{\log^4\widetilde{T}}{\widetilde{T}}\sum_{q=0}^{\left[\widetilde{T}^{\varsigma}\right]-1}\ \sum_{s=1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}^{\left[\widetilde{T}^{\beta}\right]+(q+1)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}\left\{\frac{1}{|q|_+}+\frac{\widetilde{T}^{(1-\varsigma)/2}}{|q|_+^{3/2}}\right\} \\
\leq\ & K\frac{\log^4\widetilde{T}}{\widetilde{T}}\left(\widetilde{T}^{1-\varsigma}\log T+\widetilde{T}^{\frac{3}{2}(1-\varsigma)}\right)\leq K\widetilde{T}^{\frac{1}{2}(1-3\varsigma)}\log^4 T\leq K\widetilde{T}^{-2\delta},
\end{aligned}
$$

where in the first inequality we have used Lemma 3 and that for $q\geq 1$ and $\psi\geq 0$,

$$
\sum_{j=1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}^{s}j^{-\psi}\leq\frac{K}{\left(\widetilde{T}^{\beta}+q\widetilde{T}^{1-\varsigma}\right)^{\psi}}\left(\sum_{j=1+\left[\widetilde{T}^{\beta}\right]+q\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}^{\left[\widetilde{T}^{\beta}\right]+(q+1)\widetilde{T}/\left[\widetilde{T}^{\varsigma}\right]}1\right)\leq\frac{K\widetilde{T}^{(1-\varsigma)(1-\psi)}}{q^{\psi}}.
$$

This completes the proof. □

Remark 1 *Lemma 4 holds true for $\alpha_T^{\zeta}(\lambda)$ and $\tilde{\alpha}_T^{\zeta}(\lambda)$ replaced by*

$$
\ddot{\alpha}_T^{\zeta}(\lambda):=\alpha_T^{\zeta}(\pi)-\alpha_T^{\zeta}(\lambda),\qquad \ddot{\tilde{\alpha}}_T^{\zeta}(\lambda):=\tilde{\alpha}_T^{\zeta}(\pi)-\tilde{\alpha}_T^{\zeta}(\lambda)
$$

respectively. This is so, because the triangle inequality implies that

$$
\mathbb{E}\sup_{\lambda\in[0,\pi]}\left|\ddot{\alpha}_T^{\zeta}(\lambda)-\ddot{\tilde{\alpha}}_T^{\zeta}(\lambda)\right|\leq 2\mathbb{E}\sup_{\lambda\in[0,\pi]}\left|\alpha_T^{\zeta}(\lambda)-\tilde{\alpha}_T^{\zeta}(\lambda)\right|.
$$

Define for μ and $\vartheta \in [0, \pi]$,

$$c_s(\mu, \vartheta) = \frac{2}{T\widetilde{T}^{1/2}} \sum_{p=\left[\widetilde{T}\mu/\pi\right]+1}^{\left[\widetilde{T}\vartheta/\pi\right]} \zeta_p \cos(s\lambda_p), \tag{26}$$

where ζ is as in Lemma 1 and $\mu < \vartheta$.

Lemma 5 *For $0 \leq \mu < \vartheta_1, \vartheta_2 \leq \pi$, as $T \to \infty$,*

$$\sum_{t=1}^{T-1} \sum_{s=1}^{T-t} c_s(\mu, \vartheta_1) c_s'(\mu, \vartheta_2) = g(\mu, \vartheta_1, \vartheta_2)(1 + o(1)), \tag{27}$$

where $g(\mu, \vartheta_1, \vartheta_2) = \pi^{-1} \int_\mu^{\vartheta_1 \wedge \vartheta_2} \zeta(u) \zeta'(u) du - \left(\pi^{-1} \int_\mu^{\vartheta_1} \zeta(u) du\right) \left(\pi^{-1} \int_\mu^{\vartheta_2} \zeta'(u) du\right)$.

Proof. A typical component of the matrix on the left of (27) is

$$\begin{aligned}
& \frac{4}{T^2 \widetilde{T}} \sum_{p_1 = [\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\vartheta_1/\pi]} \zeta_{p_1}^{(k_1)} \sum_{p_2 = [\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\vartheta_2/\pi]} \zeta_{p_2}^{(k_2)} \sum_{t=1}^{T-1} \sum_{s=1}^{T-t} \cos(s\lambda_{p_1}) \cos(s\lambda_{p_2}) \\
= & \frac{4}{T^2 \widetilde{T}} \sum_{p = [\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\vartheta_1/\pi] \wedge [\widetilde{T}\vartheta_2/\pi]} \zeta_p^{(k_1)} \zeta_p^{(k_2)} \sum_{t=1}^{T-1} \sum_{s=1}^{T-t} \cos^2(s\lambda_p) \qquad (28) \\
& + \frac{2}{T^2 \widetilde{T}} \sum_{p_1 = [\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\vartheta_1/\pi]} \zeta_{p_1}^{(k_1)} \sum_{p_2 = [\widetilde{T}\mu/\pi]+1; p_2 \neq p_1}^{[\widetilde{T}\vartheta_2/\pi]} \zeta_{p_2}^{(k_2)} \sum_{t=1}^{T-1} \sum_{s=1}^{T-t} \{\cos(s\lambda_{p_1+p_2}) + \cos(s\lambda_{p_1-p_2})\}.
\end{aligned}$$

Because $\cos^2 \lambda = (1 + \cos(2\lambda))/2$, then using formulae in Brillinger (1981, p. 13) we have that $\sum_{t=1}^{T-1} \sum_{s=1}^{T-t} \cos^2(s\lambda_p) = (T-1)^2/4$ and, for $p_1 \neq p_2$,

$$\sum_{t=1}^{T-1} \sum_{s=1}^{T-t} \{\cos(s\lambda_{p_1+p_2}) + \cos(s\lambda_{p_1-p_2})\} = -T,$$

and hence we conclude that the right side of (28) is, recalling that $\widetilde{T} = [T/2]$,

$$\begin{aligned}
& \frac{(T-1)^2}{T^2} \left(\frac{1}{\widetilde{T}} \sum_{p = [\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\vartheta_1/\pi] \wedge [\widetilde{T}\vartheta_2/\pi]} \zeta_p^{(k_1)} \zeta_p^{(k_2)} \right) - \frac{2}{T\widetilde{T}} \sum_{p_1 = [\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\vartheta_1/\pi]} \zeta_{p_1}^{(k_1)} \sum_{\substack{p_2 = [\widetilde{T}\mu/\pi]+1; \\ p_2 \neq p_1}}^{[\widetilde{T}\vartheta_2/\pi]} \zeta_{p_2}^{(k_2)} \\
= \; & g^{(k_1,k_2)}(\mu, \vartheta_1, \vartheta_2)(1 + o(1)),
\end{aligned}$$

by Lemma 1 and where $g^{(k_1,k_2)}(\mu, \vartheta_1, \vartheta_2)$ denotes the $(k_1, k_2)\, th$ element of the matrix $g(\mu, \vartheta_1, \vartheta_2)$. □

We now introduce the following notation. For $0 \leq v_1 < v_2 \leq \pi$,

$$\mathcal{E}_{1,T}(v_1, v_2) := \left(\frac{1}{\widetilde{T}} \sum_{p = [\widetilde{T}v_1/\pi]+1}^{[\widetilde{T}v_2/\pi]} \zeta_p \right) \left(\frac{\widetilde{T}^{1/2}}{T} \sum_{t=1}^{T} \left(\varepsilon^2(t) - \sigma^2\right) \right) \tag{29}$$

$$\mathcal{E}_{2,T}(v_1, v_2) := \sum_{t=2}^{T} \varepsilon(t) \sum_{s=1}^{t-1} \varepsilon(s) c_{t-s}(v_1, v_2), \tag{30}$$

where $c_t(\cdot, \cdot)$ is given in (26) and ζ is as in Lemma 1.

Lemma 6 *Let $0 \leq v_1 < v < v_2 < \pi$. Then assuming A1, for $k = 1, ..., p$ and for some $\beta > 0$ and $0 < \delta < 1$,*

$$\mathbb{E}\left(\left|\mathcal{E}_{j,T}^{(k)}(v_1, v)\right|^{\beta}\left|\mathcal{E}_{j,T}^{(k)}(v, v_2)\right|^{\beta}\right) \leq K(v_2 - v_1)^{2-\delta}, \; j = 1, 2, \tag{31}$$

where $\mathcal{E}_{1,T}^{(k)}(v_1, v)$ and $\mathcal{E}_{2,T}^{(k)}(v_1, v)$ are the kth components of (29) and (30) respectively.

Proof. We begin with $j = 1$. By Lemma 1,

$$\left|\frac{1}{\widetilde{T}}\sum_{p=\left[\widetilde{T}v_1/\pi\right]+1}^{\left[\widetilde{T}v_2/\pi\right]}\zeta_p^{(k)} - \frac{1}{\pi}\int_{v_1}^{v_2}\zeta^{(k)}(x)\,dx\right| \leq K\frac{\left|\log\widetilde{T}\right|^{\ell}}{\widetilde{T}} \leq K(v_2 - v_1)^{1-\delta/2},$$

after we notice that we can take $\widetilde{T}^{-1} \leq (v_2 - v_1)$, since otherwise (31) holds trivially. On the other hand, $A1$ implies that $\mathbb{E}\left(\sum_{t=1}^{T}\left(\varepsilon^2(t) - \sigma^2\right)\right)^2 \leq KT$. So, using the inequality $(v_2 - v)(v - v_1) < (v_2 - v_1)^2$ and Cauchy-Schwarz inequality, we have that $\mathbb{E}\left(\left|\mathcal{E}_{1,T}^{(k)}(v_1, v)\right|\left|\mathcal{E}_{1,T}^{(k)}(v, v_2)\right|\right) \leq K(v_2 - v_1)^{2-\delta}$.

To complete the proof, it suffices to examine that the inequality in (31) holds true for $j = 2$. Now

$$\mathbb{E}\left(\mathcal{E}_{2,T}^{(k)}(v_1, v_2)\right)^4 = 16\prod_{j=1}^{4}\sum_{1\leq s_j<t_j\leq T}c_{t_j-s_j}^{(k)}(v_1, v_2)\,\mathbb{E}\left(\varepsilon(t_1)\varepsilon(s_1)...\varepsilon(t_4)\varepsilon(s_4)\right).$$

Since the number of equal indices in the set $\{t_1, s_1, ..., t_4, s_4\}$ does not exceed 4, by Assumption $A1$, it follows that $\left|\mathbb{E}\left(\varepsilon(t_1)\varepsilon(s_1)...\varepsilon(t_4)\varepsilon(s_4)\right)\right| \leq K$. Moreover, by $A1$, the inequality $\left|\mathbb{E}\left(\varepsilon(t_1)\varepsilon(s_1)...\varepsilon(t_4)\varepsilon(s_4)\right)\right| \neq 0$ can hold only if any t_j, s_j are repeated in $\{t_1, s_1, ..., t_4, s_4\}$ at least twice. Hence by Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}_{2,T}^{(k)}(v_1, v_2)\right)^4 &\leq K\prod_{j=1}^{4}\left(\sum_{1\leq s_j<t_j\leq T}\left(c_{t_j-s_j}^{(k)}(v_1, v_2)\right)^2\right)^{1/2} \\ &= K\left(\sum_{1\leq s<t\leq T}\left(c_{t-s}^{(k)}(v_1, v_2)\right)^2\right)^2.\end{aligned}$$

But by Lemma 5, the right side of the last displayed equation is bounded by

$$K\left(\frac{1}{\pi}\int_{v_1}^{v_2}\left(\zeta^{(k)}(u)\right)^2 du - \left(\frac{1}{\pi}\int_{v_1}^{v_2}\zeta^{(k)}(u)\,du\right)^2\right)^2 \leq K(v_2 - v_1)^{2-\delta}$$

because $\left|\int_{v_1}^{v_2}\left(\zeta^{(k)}(x)\right)^p dx\right| \leq K\left|v_2 - v_1\right|^{1-\delta/2}$ for $p = 1, 2$. This concludes the proof choosing $\beta = 2$ by the Cauchy-Schwarz's inequality. □

Lemma 7 *Denote* $\eta_p := I_{\varepsilon,p} - \sigma^2 / (2\pi)$ *and*

$$R_T^1(v) = \frac{2\pi}{\widetilde{T}^{1/2}} \sum_{p=1}^{[\widetilde{T}v/\pi]} \zeta_p \eta_p \quad \text{and } R_T^2(v) = \frac{2\pi}{\widetilde{T}^{1/2}} \sum_{p=[\widetilde{T}v/\pi]+1}^{\widetilde{T}} \zeta_p \eta_p, \quad (0 \le v < \pi)$$

with ζ *as in Lemma 1. Let* $0 \le v_1 < v < v_2 \le \pi$. *Then assuming A1, for some* $\beta > 0$ *and* $0 < \delta < 1$,

$$(a)\ \mathbb{E}\left(\left\| R_T^j(v_2) - R_T^j(v) \right\|^{\beta} \left\| R_T^j(v) - R_T^j(v_1) \right\|^{\beta} \right) \le K (v_2 - v_1)^{2-\delta},\ j = 1, 2. \tag{32}$$

$$(b)\ R_T^j(v) \xrightarrow{d} \mathcal{N}\left(0, 4\pi^2 V^{(j)}(v)\right),\ j = 1, 2,$$

where $V^{(1)}(v) = \sigma^4 \int_0^v \zeta(u) \zeta'(u)\, du/\pi + \sigma^4 \kappa \int_0^v \zeta(u)\, du \int_0^v \zeta'(u)\, du/\pi^2$ *and* $V^{(2)}(v) = \sigma^4 \int_v^\pi \zeta(u) \zeta'(u)\, du/\pi + \sigma^4 \kappa \int_v^\pi \zeta(u)\, du \int_v^\pi \zeta'(u)\, du/\pi^2$, *with* κ *denoting the fourth cumulant of* $\{\varepsilon(t)/\sigma\}_{t\in\mathbb{Z}}$.

Proof. We begin with (a). We shall consider $R_T^2(v)$ only, $R_T^1(v)$ being similarly handled. From the definition of η_p, and that

$$R_T^2(v) - R_T^2(v_2) = \frac{2\pi}{\widetilde{T}^{1/2}} \sum_{p=[\widetilde{T}v/\pi]+1}^{[\widetilde{T}v_2/\pi]} \zeta_p \eta_p,$$

we have that

$$R_T^2(v) - R_T^2(v_2) = \mathcal{E}_{1,T}(v, v_2) + \mathcal{E}_{2,T}(v, v_2),$$

where $\mathcal{E}_{1,T}(v, v_2)$ and $\mathcal{E}_{2,T}(v, v_2)$ are given in (29) and (30) respectively. Now (32) follows immediately from Lemma 6 and standard inequalities.

Part (b). We will examine $R_T^1(v) \xrightarrow{d} \mathcal{N}\left(0, 4\pi^2 V^{(1)}(v)\right)$, being the proof for $j = 2$ identically handled. But this follows by an obvious extension of Theorem 4.2 of Giraitis, Hidalgo and Robinson (2001) because $\zeta(u)$ satisfies the same conditions of $h_n(u)$ there. $\square$

Lemma 8 *Assume A1 − A4. Then, we have that for some* $0 < \delta < 1/6$,

$$\begin{aligned}(a)\ \frac{2\pi}{\widetilde{T}^{1/2}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j \left(\frac{I_{X,j}}{h_{\theta_T,j}} - \frac{\sigma^2}{2\pi} \right) &= \frac{2\pi}{\widetilde{T}^{1/2}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j \left(I_{\varepsilon,j} - \frac{\sigma^2}{2\pi} \right) \\ &\quad - \left(\frac{\sigma^2}{\widetilde{T}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j \phi'_{\theta_0,j} \right) \widetilde{T}^{1/2} (\theta_T - \theta_0) \\ &\quad + O_p\left(\frac{1}{T^\delta} \right),\end{aligned} \tag{33}$$

$$(b)\ \frac{2\pi}{\widetilde{T}^{1/2}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\zeta_j\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right) = \frac{2\pi}{\widetilde{T}^{1/2}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\zeta_j\left(I_{\varepsilon,j}-\frac{\sigma^2}{2\pi}\right) - \left(\frac{\sigma^2}{\widetilde{T}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\zeta_j\phi'_{\theta_0,j}\right)\widetilde{T}^{1/2}\left(\theta_T-\theta_0\right) + O_p\left(\frac{1}{T^{\delta}}\right),$$

where the $O_p\left(1/T^{\delta}\right)$ is uniform in $\lambda\in[0,\pi]$, and where $\zeta(u)$ and $\|\zeta(u)\|$ are as in Lemma 1.

Proof. We examine (a), part (b) being similarly handled. The difference between the left side of (33) and the first term on its right side is

$$\frac{2\pi}{\widetilde{T}^{1/2}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}\zeta_j\frac{I_{X,j}}{h_{\theta_0,j}}\left[\frac{h_{\theta_0,j}}{h_{\theta_T,j}}-1+\phi'_{\theta_0,j}\left(\theta_T-\theta_0\right)\right] \quad (34)$$
$$+\frac{2\pi}{\widetilde{T}^{1/2}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}\zeta_j\left(\frac{I_{X,j}}{h_{\theta_0,j}}-I_{\varepsilon,j}\right)-\frac{2\pi}{\widetilde{T}^{1/2}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}\zeta_j\phi'_{\theta_0,j}\frac{I_{X,j}}{h_{\theta_0,j}}\left(\theta_T-\theta_0\right).$$

First we notice that

$$\theta_T-\theta_0=O_p\left(T^{-1/2}\right), \quad (35)$$

which follows by (8) in Assumption $A4$, and because

$$\frac{1}{\widetilde{T}^{1/2}}\sum_{k=1}^{\widetilde{T}}\phi_{\theta_0,k}\left(\frac{I_{X,k}}{h_{\theta_0,k}}-I_{\varepsilon,k}\right)=O_p\left(T^{-\delta}\right), \quad (36)$$

(recall that under H_0, $h_j=h_{\theta_0,j}$), by Lemma 4 and Markov's inequality, and

$$\frac{2\pi}{\sigma^2\widetilde{T}^{1/2}}\sum_{k=1}^{\widetilde{T}}\phi_{\theta_0,k}I_{\varepsilon,k}\xrightarrow{d}\mathcal{N}\left(0,\frac{1}{\pi}\int_0^{\pi}\phi_{\theta_0}(u)\,\phi'_{\theta_0}(u)\,du\right)\overset{d}{=}\int_0^{\pi}\phi_{\theta_0}(u)\,B_{\pi}(du) \quad (37)$$

by Lemma 7 with $\zeta(u)=\phi_{\theta_0}(u)$. Notice also that $\sum_{k=1}^{\widetilde{T}}\phi_{\theta_0,k}=O(\log T)$ by Lemma 1 because (9) and that $A3$ part (c) implies that $\phi_{\theta_0}(\lambda)$ satisfies the same conditions of $\zeta(\lambda)$ in Lemma 1.

Next, $A3$ part (d) implies that, uniformly in $\lambda\in[0,\pi]$, the norm of the first term of (34) is bounded by

$$K\widetilde{T}^{1/2}\left\|\theta_T-\theta_0\right\|^2\frac{1}{\widetilde{T}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}\left|\log^2\lambda_j\right|\left\|\zeta_j\right\|\frac{I_{X,j}}{h_{\theta_0,j}}=O_p\left(T^{-1/2}\right), \quad (38)$$

because (35) implies that we can take $\delta = KT^{-1/2}$ in $A3$ part (d) so that $\lambda_j^{-\delta} < K$ when $\delta < KT^{-1/2}$ and $j \geq 1$, and also because by Markov's inequality and Lemmas 4 and 7,

$$\sup_{\lambda\in[0,\pi]} \left| \frac{1}{\widetilde{T}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \left|\log^2 \lambda_j\right| \left\|\zeta_j\right\| \left(\frac{I_{X,j}}{h_{\theta_0,j}} - \frac{\sigma^2}{2\pi} \right) \right| = O_p\left(T^{-1/2}\right)$$

and because by Lemma 1 with $\|\zeta(u)\| \left|\log^2(u)\right|$ there,

$$\sup_{\lambda\in[0,\pi]} \left| \frac{1}{\widetilde{T}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \left|\log^2 \lambda_j\right| \left\|\zeta_j\right\| - \frac{1}{\pi} \int_0^\lambda \left|\log^2(u)\right| \|\zeta(u)\| \, du \right| = o\left(\widetilde{T}^{-1/2}\right).$$

The second term of (34) is $O_p\left(T^{-\delta}\right)$ by Lemma 4 and Markov's inequality. Next, proceeding similarly as in (38), since $\zeta(\lambda)\,\phi'_{\theta_0}(\lambda)$ satisfies the same conditions of $\zeta(\lambda)\left|\log \lambda\right|$, the third term of (34) is $\widetilde{T}^{-1}\sigma^2 \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} \zeta_j \phi'_{\theta_0,j} \widetilde{T}^{1/2}(\theta_T - \theta_0) + O_p\left(T^{-\delta}\right)$, which concludes the proof.$\square$

Lemma 9 *Assuming A1, for any $0 \leq \upsilon < (1-\delta)/4$, with δ as in Lemma 7, we have that for all $k = 1, ..., p$,*

$$(a)\ \mathbb{E}\left(\frac{\mathcal{E}_{1,T}^{(k)}(\lambda_1,\pi)}{(\pi-\lambda_1)^\upsilon} - \frac{\mathcal{E}_{1,T}^{(k)}(\lambda_2,\pi)}{(\pi-\lambda_2)^\upsilon} \right)^2 \leq K(\lambda_2-\lambda_1)^{2-\delta-2\upsilon} \tag{39}$$

$$(b)\ \mathbb{E}\left(\frac{\mathcal{E}_{2,T}^{(k)}(\lambda_1,\pi)}{(\pi-\lambda_1)^\upsilon} - \frac{\mathcal{E}_{2,T}^{(k)}(\lambda_2,\pi)}{(\pi-\lambda_2)^\upsilon} \right)^4 \leq K(\lambda_2-\lambda_1)^{2-\delta-4\upsilon} \tag{40}$$

for all $0 < \lambda_1 < \lambda_2 < \pi$, and where $\mathcal{E}_{1,T}^{(k)}(\lambda_1,\lambda_2)$ and $\mathcal{E}_{2,T}^{(k)}(\lambda_1,\lambda_2)$ are given in (29) *and* (30) *respectively.*

Proof. We begin with (b). By standard inequalities, the left side of (40) is bounded by

$$K\mathbb{E}\left(\frac{1}{(\pi-\lambda_1)^\upsilon} \mathcal{E}_{2,T}^{(k)}(\lambda_1,\lambda_2) \right)^4 + K\left(\frac{1}{(\pi-\lambda_1)^\upsilon} - \frac{1}{(\pi-\lambda_2)^\upsilon} \right)^4 \mathbb{E}\left(\mathcal{E}_{2,T}^{(k)}(\lambda_2,\pi)\right)^4.$$

By Lemma 6, for any $0 < \delta < 1$, we have that the last displayed expression is bounded by

$$K\frac{(\lambda_2-\lambda_1)^{2-\delta}}{(\pi-\lambda_1)^{4\upsilon}} + K\left(\frac{1}{(\pi-\lambda_1)^\upsilon} - \frac{1}{(\pi-\lambda_2)^\upsilon} \right)^4 (\pi-\lambda_2)^{2-\delta}. \tag{41}$$

Consider the case that $\lambda_2 - \lambda_1 \leq 2^{-1}(\pi - \lambda_2)$ first. By mean value theorem, (41) is

$$\begin{aligned} & K\frac{(\lambda_2-\lambda_1)^{2-\delta}}{(\pi-\lambda_1)^{4\upsilon}} + \frac{K}{(\pi-\lambda_1)^{4\upsilon}(\pi-\lambda_2)^{\delta+4\upsilon-2}} \frac{\upsilon^4(\lambda_2-\lambda_1)^4}{(\beta(\pi-\lambda_1)+(1-\beta)(\pi-\lambda_2))^{4-4\upsilon}} \\ \leq\ & K(\lambda_2-\lambda_1)^{2-\delta-4\upsilon} + K(\pi-\lambda_2)^{-\delta-4\upsilon-2}(\lambda_2-\lambda_1)^4 \end{aligned}$$

where $\beta = \beta(\lambda_1, \lambda_2) \in (0,1)$, and then because $\pi - \lambda_1 > \lambda_2 - \lambda_1$ and $\pi - \lambda_1 \geq \pi - \lambda_2 > 0$. But the right side of the last displayed inequality is bounded by $K(\lambda_2 - \lambda_1)^{2-\delta-4\upsilon}$ since $\lambda_2 - \lambda_1 \leq 2^{-1}(\pi - \lambda_2)$.

Next, consider the case for which $2^{-1}(\pi - \lambda_2) < \lambda_2 - \lambda_1$. Using the inequality $a^{\varsigma} - b^{\varsigma} \leq (a-b)^{\varsigma}$ for any $0 < \varsigma < 1$ and $a \geq b$, we have that (41) is bounded by

$$K(\lambda_2 - \lambda_1)^{2-\delta-4\upsilon} + K\frac{(\lambda_2 - \lambda_1)^{4\upsilon}(\pi - \lambda_2)^{2-\delta}}{(\pi - \lambda_1)^{4\upsilon}(\pi - \lambda_2)^{4\upsilon}} \leq K(\lambda_2 - \lambda_1)^{2-\delta-4\upsilon},$$

where we have used that $0 < \lambda_2 - \lambda_1 \leq \pi - \lambda_1$ and $\pi - \lambda_2 < 2(\lambda_2 - \lambda_1)$. This completes the proof of part (b).

Next part (a). By definition and $A1$, the left side of (39) is bounded by

$$\begin{aligned} & \frac{K}{(\pi - \lambda_1)^{2\upsilon}} \left(\frac{1}{\widetilde{\widetilde{T}}} \sum_{j=[\widetilde{T}\lambda_1/\pi]+1}^{[\widetilde{T}\lambda_2/\pi]} \zeta_j^{(k)} \right)^2 \\ & + K\left(\frac{1}{(\pi - \lambda_1)^{\upsilon}} - \frac{1}{(\pi - \lambda_2)^{\upsilon}} \right)^2 \left(\frac{1}{\widetilde{\widetilde{T}}} \sum_{j=[\widetilde{T}\lambda_2/\pi]+1}^{\widetilde{T}} \zeta_j^{(k)} \right)^2 \\ \leq \; & K(\lambda_2 - \lambda_1)^{2-\delta-2\upsilon} \end{aligned}$$

by Lemma 1 and proceeding as in part (b). □

In what follows we shall abbreviate $\gamma'_{\theta,q} A^{-1}_{\theta,T}(q)$ by $H_{\theta,T}(q)$.

Lemma 10 *Assuming $A1 - A5$, for all $\varepsilon > 0$,*

$$\lim_{\lambda_0 \to \pi} \limsup_{T \to \infty} \Pr\left\{ \sup_{\lambda_0 \leq \lambda \leq \pi} \left| \frac{1}{\widetilde{\widetilde{T}}} \sum_{k=[\widetilde{T}\lambda_0/\pi]+1}^{[\widetilde{T}\lambda/\pi]} \frac{H_{\theta_0,T}(k)}{\widetilde{T}^{1/2}} \sum_{j=k+1}^{\widetilde{T}} \gamma_{\theta_0,j} \left(\frac{I_{X,j}}{h_{\theta_T,j}} - \frac{\sigma^2}{2\pi} \right) \right| > \varepsilon \right\} = 0. \tag{42}$$

Proof. Abbreviate $h_{\theta_T,j}^{-1} I_{X,j} - I_{\varepsilon,j}$ by $\varkappa_j$ and take $\lambda_0 > \pi/2$ without loss of generality. Noting that $h_{\theta_T,j}^{-1} I_{X,j} - \sigma^2/(2\pi) = \varkappa_j + \eta_j$, where $\eta_j = I_{\varepsilon,j} - \sigma^2/(2\pi)$, we have that

$$\sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{T}} \sum_{k=[\widetilde{T}\lambda_0/\pi]+1}^{[\widetilde{T}\lambda/\pi]} \frac{H_{\theta_0,T}(k)}{\widetilde{T}^{1/2}} \sum_{j=k+1}^{\widetilde{T}} \gamma_{\theta_0,j} \left(\varkappa_j + \eta_j\right) \right| \tag{43}$$

$$\le \frac{K}{\widetilde{T}} \sum_{k=[\widetilde{T}\lambda_0/\pi]+1}^{\overline{T}} \left\| H_{\theta_0,T}(k) \right\| \left(1 - \frac{k}{\widetilde{T}}\right)^{\frac{\delta}{2}} \left\{ \sup_{[\widetilde{T}\lambda_0/\pi] \le k \le \overline{T}} \left\| \frac{\left(1 - \frac{k}{\widetilde{T}}\right)^{-\frac{\delta}{2}}}{\widetilde{T}^{1/2}} \sum_{j=k+1}^{\widetilde{T}} \gamma_{\theta_0,j} \varkappa_j \right\| \right.$$

$$\left. + \sup_{[\widetilde{T}\lambda_0/\pi] \le k \le \overline{T}} \left\| \frac{\left(1 - \frac{k}{\widetilde{T}}\right)^{-\frac{\delta}{2}}}{\widetilde{T}^{1/2}} \sum_{j=k+1}^{\widetilde{T}} \gamma_{\theta_0,j} \eta_j \right\| \right\},$$

for any $0 < \delta < 1$. The first factor on the right of (43) is bounded by

$$K \left| \frac{1}{\widetilde{T}} \sum_{k=[\widetilde{T}\lambda_0/\pi]+1}^{\overline{T}} \left\| \gamma_{\theta_0,k} \right\| \left(1 - \frac{k}{\widetilde{T}}\right)^{\frac{\delta}{2}-1} \right| \le K \left(\frac{\overline{T} - [\widetilde{T}\lambda_0/\pi]}{\widetilde{T}} \right)^{\frac{\delta}{2}},$$

using that

$$\left\| A_{\theta_0,T}^{-1}(k) \right\| \le K \left(1 - \frac{k}{\widetilde{T}}\right)^{-1},$$

because $\left\| A_{\theta_0}(\lambda) \right\| \ge K^{-1}(\pi - \lambda)$ by Assumption $A5$ and because Lemma 1 implies that $\sup_{[\widetilde{T}\lambda_0/\pi] \le k \le \overline{T}} \left\| A_{\theta_0,T}(k) - A_{\theta_0}\left(\left[k\pi/\widetilde{T}\right]\right) \right\| = O\left(T^{-1}\log^2 T\right)$.

Next, by Lemma 9, the second term inside the braces on the right of (43) is $O_p(1)$ for $\delta > 0$ small enough, whereas Lemma 8 and (35) imply that the first term is bounded by

$$\sup_{[\widetilde{T}\lambda_0/\pi] \le k \le \overline{T}} \left\| \frac{\left(1 - \frac{k}{\widetilde{T}}\right)^{-\frac{\delta}{2}}}{\widetilde{T}} \sum_{j=k+1}^{\widetilde{T}} \gamma_{\theta_0,j} \phi'_{\theta_0,j} \right\| O_p(1) + O_p\left(\sup_{[\widetilde{T}\lambda_0/\pi] \le k \le \overline{T}} \frac{\left(1 - \frac{k}{\widetilde{T}}\right)^{-\frac{\delta}{2}}}{T^{\delta}} \right)$$

$$= O_p\left(\left|\pi - \lambda_0\right|^{\frac{\delta}{2}} \right)$$

because $T^{-1} \le \widetilde{T}^{-1} \le \inf_{[\widetilde{T}\lambda_0/\pi] \le k \le \overline{T}} \left(1 - k/\widetilde{T}\right)$, $0 < \delta < 1$ and an obvious extension of Lemma 1 but with $\zeta(\lambda) = \gamma_{\theta_0}(\lambda) \phi'_{\theta_0}(\lambda)$ there. So, (43) is $O_p\left(\left|\pi - \lambda_0\right|^{\delta}\right)$, which implies that (42) holds true because $\delta > 0$. $\square$

Lemma 11 *Assuming* $A1 - A6$,

$$\sup_{\lambda \in [0,\pi]} \left\| \frac{1}{\widetilde{T}^{1/2}} \sum_{j=[\widetilde{T}\lambda/\pi]+1}^{\widetilde{T}} \left(\phi_{\theta_T,j} - \phi_{\theta_0,j}\right) \left(\frac{I_{X,j}}{h_{\theta_T,j}} - \frac{\sigma^2}{2\pi} \right) \right\| = O_p\left(\frac{\log T}{T^{1/2}} \right). \tag{44}$$

Proof. The expression inside the norm on the left of (44) is

$$
\begin{aligned}
&\frac{1}{\widetilde{T}^{1/2}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\dot{\phi}_{\theta_0,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-I_{\varepsilon,j}\right)(\theta_T-\theta_0)\\
&+\frac{1}{\widetilde{T}^{1/2}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\dot{\phi}_{\theta_0,j}\left(I_{\varepsilon,j}-\frac{\sigma^2}{2\pi}\right)(\theta_T-\theta_0)\\
&+\frac{1}{\widetilde{T}^{1/2}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\left(\phi_{\theta_T,j}-\phi_{\theta_0,j}-\dot{\phi}_{\theta_0,j}(\theta_T-\theta_0)\right)\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right). \qquad (45)
\end{aligned}
$$

By $A6$ and then noting that $|a-b|\leq(a-b)+2b$ for $a>0$ and $b>0$, the norm of the third term of (45) is bounded by

$$
\begin{aligned}
& K\frac{\|\theta_T-\theta_0\|^2}{\widetilde{T}^{1/2}}\sum_{j=1}^{\widetilde{T}}|\log(\lambda_j)|\left|\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right|\\
\leq\; & K\frac{\|\theta_T-\theta_0\|^2}{\widetilde{T}^{1/2}}\left\{\sum_{j=1}^{\widetilde{T}}|\log(\lambda_j)|\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right)+\frac{\sigma^2}{\pi}\sum_{j=1}^{\widetilde{T}}|\log\lambda_j|\right\}=O_p\left(\frac{\log T}{T^{1/2}}\right)
\end{aligned}
$$

by (35) and then using Lemmas 8 and 7 with $\zeta(\lambda)=|\log\lambda|$, and Lemma 1 respectively. So, uniformly in λ the third term of (45) is $o_p(1)$. Likewise, the first term of (45) is $O_p\left(T^{-1/2}\right)$ uniformly in λ using Lemma 8 with $\zeta(\lambda)=\dot{\phi}_{\theta_0}(\lambda)$ and (35). Observe that $\dot{\phi}_{\theta_0}(\lambda)$ satisfies the same conditions that $\zeta(\lambda)$ in Lemma 8 by $A6$. Finally, the second term of (45) is $O_p\left(T^{-1/2}\right)$ by Lemma 7 with $\zeta(\lambda)=\dot{\phi}_{\theta_0}(\lambda)$. □

Lemma 12 *Assuming $A1-A6$, for all $\varepsilon>0$,*

$$
\lim_{\lambda_0\to\pi}\limsup_{T\to\infty}\Pr\left\{\sup_{\lambda_0\leq\lambda\leq\pi}\left|\frac{1}{\widetilde{T}}\sum_{q=\left[\widetilde{T}\lambda_0/\pi\right]+1}^{\left[\widetilde{T}\lambda/\pi\right]}\frac{H_{\theta_T,T}(q)}{\widetilde{T}^{1/2}}\sum_{j=q+1}^{\widetilde{T}}\gamma_{\theta_T,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right)\right|>\varepsilon\right\}=0. \qquad (46)
$$

Proof. Notice that (35) implies that it suffices to show (46) in the set $\left\{\|\theta_T-\theta_0\|<KT^{-1/2}m_T^{-1}\right\}$, where $m_T+m_T^{-1}T^{-1/2}\to 0$. On the other hand, Lemma 11 and then Lemma 8 imply that, uniformly in q,

$$
\begin{aligned}
\frac{1}{\widetilde{T}^{1/2}}\sum_{j=q+1}^{\widetilde{T}}\gamma_{\theta_T,j}\varkappa_j &= \left(\frac{\sigma^2}{\widetilde{T}}\sum_{j=q+1}^{\widetilde{T}}\gamma_{\theta_0,j}\phi'_{\theta_0,j}\right)\widetilde{T}^{1/2}(\theta_0-\theta_T)+O_p\left(T^{-\delta}\right)\\
\frac{1}{\widetilde{T}^{1/2}}\sum_{j=q+1}^{\widetilde{T}}\gamma_{\theta_T,j}\eta_j &= \frac{1}{\widetilde{T}^{1/2}}\sum_{j=q+1}^{\widetilde{T}}\gamma_{\theta_0,j}\eta_j+O_p\left(T^{-1/2}\right) \qquad (47)
\end{aligned}
$$

proceeding as in the proof of (44) but with $\varkappa_j + \eta_j$ replaced by η_j there. Observe that we can take $\lambda_0 > \pi/2$. Next, uniformly in q, $A6$ implies that

$$\sup_{\left[\overline{T}\lambda_0/\pi\right] \le q \le \overline{T}} \left\| A_{\theta_T,T}(q) - A_{\theta_0,T}(q) \right\| = (\pi - \lambda_0)\, O_p\left(\|\theta_T - \theta_0\|\right)$$

which will imply that, with probability approaching one, as $T \to \infty$,

$$\left\| A^{-1}_{\theta_T,T}(q) \right\| \le \left\| A^{-1}_{\theta_0,T}(q) \right\| \left(1 + KT^{-1/2} m_T^{-1}\right) \le K \left(1 - \frac{q}{\widetilde{T}}\right)^{-1},$$

because $\left\| A_{\theta_0}(\lambda) \right\| \ge K^{-1}(\pi - \lambda)$ and Lemma 1 implies that $\sup_{\left[\overline{T}\lambda_0/\pi\right] \le q \le \overline{T}} \left\| A_{\theta_0,T}(q) - A_{\theta_0}\left(\left[q\pi/\widetilde{T}\right]\right) \right\| = O\left(T^{-1}\log^2 T\right)$. So, we have that for $0 < \delta < 1/2$,

$$\sup_{\lambda_0 \le \lambda \le \pi} \left\| \frac{1}{\widetilde{T}} \sum_{q=\left[\overline{T}\lambda_0/\pi\right]+1}^{\left[\overline{T}\lambda/\pi\right]} \frac{H_{\theta_T,T}(q)}{\widetilde{T}^{1/2}} \sum_{j=q+1}^{\widetilde{T}} \gamma_{\theta_T,j} \left(\frac{I_{X,j}}{h_{\theta_T,j}} - \frac{\sigma^2}{2\pi}\right) \right\| \tag{48}$$

$$\begin{aligned}
\le\ & K \sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{T}} \sum_{q=\left[\overline{T}\lambda_0/\pi\right]+1}^{\left[\overline{T}\lambda/\pi\right]} \left\| \gamma_{\theta_0,q} \right\| \left(1 - \frac{q}{\widetilde{T}}\right)^{-1+\delta/2} \right| \\
& \times \left\{ \sup_{\left[\overline{T}\lambda_0/\pi\right] \le q \le \overline{T}} \left\| \left(1 - \frac{q}{\widetilde{T}}\right)^{-\delta/2} \frac{1}{\widetilde{T}^{1/2}} \sum_{j=q+1}^{\widetilde{T}} \gamma_{\theta_0,j}\eta_j \right\| + O_p\left(|\pi - \lambda_0|^{\delta/2}\right) \right\},
\end{aligned}$$

by (47) and because $T^{-1} \le \widetilde{T}^{-1} \le \inf_{\left[\overline{T}\lambda_0/\pi\right] \le q \le \overline{T}} \left(1 - q/\widetilde{T}\right)$. But Lemma 9 implies that $\sup_{\left[\overline{T}\lambda_0/\pi\right] \le q \le \overline{T}} \left\| \left(1 - q/\widetilde{T}\right)^{-\delta/2} \widetilde{T}^{-1/2} \sum_{j=q+1}^{\widetilde{T}} \gamma_{\theta_0,j}\eta_j \right\| = O_p(1)$, and $A3$ implies that

$$\sup_{\lambda_0 \le \lambda \le \pi} \frac{1}{\widetilde{T}} \sum_{q=\left[\overline{T}\lambda_0/\pi\right]+1}^{\left[\overline{T}\lambda/\pi\right]} \left\| \gamma_{\theta_0,q} \right\| \left(1 - \frac{q}{\widetilde{T}}\right)^{-1+\delta/2} \le K \left(\frac{\overline{T} - \left[\overline{T}\lambda_0/\pi\right]}{\widetilde{T}}\right)^{\delta/2},$$

and hence the left side of (48) is $O_p\left(|\pi - \lambda_0|^{\delta/2}\right)$. From here we conclude that (46) holds true because $\delta > 0$. $\square$

7. PROOFS

This section provides the proofs of the main results which are based on the series of lemmas given in the previous section.

Proof of Theorem 1

Part (a) follows by Lemma 4 with $\zeta(\lambda) = 1$ there. The proof of part (b) follows immediately from part (a) and Lemma 7 with $\zeta(\lambda) = 1$ there. $\square$

Proof of Theorem 2

Part (a). By Lemma 8 with $\zeta(\lambda) = 1$ there and the definitions of $G_{\theta,T}(\lambda)$ and $G_T^0(\lambda)$, we have that

$$\begin{aligned}
\widetilde{T}^{1/2}\left(G_{\theta_T,T}(\lambda) - G_T^0(\lambda)\right) &= -\left(\frac{\sigma^2}{\widetilde{T}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\phi'_{\theta_0,j}\right)\widetilde{T}^{1/2}(\theta_T - \theta_0) + o_p(1) \qquad (49)\\
&= -\left(\frac{\sigma^2}{\widetilde{T}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\phi'_{\theta_0,j}\right)S_T^{-1}\frac{2\pi}{G_{\theta_0,T}(\pi)\,\widetilde{T}^{1/2}}\sum_{k=1}^{\widetilde{T}}\phi_{\theta_0,k}\frac{I_{X,k}}{h_{\theta_0,k}}\\
&\quad + o_p(1),
\end{aligned}$$

by (8) and (9), and where the $o_p(1)$ is uniform in $\lambda \in [0,\pi]$. Likewise,

$$\widetilde{T}^{1/2}\left(G_{\theta_T,T}(\pi) - G_T^0(\pi)\right) = o_p(1) \qquad (50)$$

because (36) and (37) and that by Lemma 1 with $\zeta(\lambda) = \phi_{\theta_0}(\lambda)$ and (9), we have that $\left\|\widetilde{T}^{-1}\sum_{j=1}^{\widetilde{T}}\phi_{\theta_0,j}\right\| = O\left(T^{-1}\log T\right)$. So, (50) holds true. Also, it is worth noticing that Lemma 1 with $\zeta(\lambda) = \phi_{\theta_0}(\lambda)\,\phi'_{\theta_0}(\lambda)$ implies that $\|S_T - \Sigma_{\theta_0}\| = O\left(T^{-1}\log^2 T\right)$.

On the other hand, noting that (50) and $A1$ imply that

$$G_T^0(\pi) = \sigma^2 + O_p\left(T^{-1/2}\right), \qquad (51)$$

and that $\left|G_{\theta_0,T}(\pi) - G_T^0(\pi)\right| = o_p\left(\widetilde{T}^{-1/2}\right)$ by Lemma 4, then by $(49), (50)$ and (36), uniformly in λ, we obtain that

$$\begin{aligned}\alpha_{\theta_T,T}(\lambda) &= \alpha_T^0(\lambda) + \frac{\widetilde{T}^{1/2}\left(G_{\theta_T,T}(\lambda) - G_T^0(\lambda)\right)}{G_T^0(\pi)} \\ &\quad + G_{\theta_T,T}(\lambda)\,\widetilde{T}^{1/2}\left(\frac{1}{G_{\theta_T,T}(\pi)} - \frac{1}{G_T^0(\pi)}\right) \\ &= \alpha_T^0(\lambda) - \frac{1}{\widetilde{\widetilde{T}}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\left[\phi_{\theta_0,j}' S_T^{-1}\frac{2\pi}{G_T^0(\pi)\,\widetilde{T}^{1/2}}\sum_{k=1}^{\widetilde{T}}\phi_{\theta_0,k}I_{\varepsilon,k}\right] + o_p(1),\end{aligned} \tag{52}$$

which concludes the proof of part (a).

Next part (b). Taking into account part (a), part (b) follows because Lemma 7 guarantees the fidi's convergence of α_T^0 and its tightness. Tightness of the second term on the right of (52) follows by (37) and Lemma 1 and then because $\int_0^\lambda \phi_{\theta_0}(u)du$ is Hölder's continuous of order greater than 1/2 by $A3$. This concludes the proof of the theorem. □

Proof of Theorem 3

Using (51) and recalling that $H_{\theta,T}(j) = \gamma_{\theta,j}' A_{\theta,T}^{-1}(j)$, we obtain that

$$\beta_T^0(\lambda) = \frac{1}{\widetilde{T}^{1/2}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\left(\left(\frac{2\pi}{\sigma^2}I_{\varepsilon,j} - 1\right) - H_{\theta_0,T}(j)\frac{1}{\widetilde{\widetilde{T}}}\sum_{k=j+1}^{\widetilde{T}}\gamma_{\theta_0,k}\left(\frac{2\pi}{\sigma^2}I_{\varepsilon,k} - 1\right)\right) + o_p(1), \tag{53}$$

where the $o_p(1)$ is uniform in $\lambda \in [0,\pi]$.

Suppose, to be shown later, that the convergence in $[0,\lambda_0]$ holds true for any $0 < \lambda_0 < \pi$. Then, because B_π and the limit of the process $\widetilde{T}^{-1/2}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\left(I_{\varepsilon,j} - \frac{\sigma^2}{2\pi}\right)$ are continuous in $[0,\pi]$, Billingsley's (1968) Theorem 4.2 implies that it suffices to show that for all $\varepsilon > 0$,

$$\lim_{\lambda_0\to\pi}\limsup_{T\to\infty}\Pr\left\{\sup_{\lambda_0\le\lambda\le\pi}\left|\frac{1}{\widetilde{\widetilde{T}}}\sum_{j=[\widetilde{T}\lambda_0/\pi]+1}^{[\widetilde{T}\lambda/\pi]}\frac{H_{\theta_0,T}(j)}{\widetilde{T}^{1/2}}\sum_{k=j+1}^{\widetilde{T}}\gamma_{\theta_0,k}\left(\frac{2\pi}{\sigma^2}I_{\varepsilon,k} - 1\right)\right| > \varepsilon\right\} = 0,$$

which follows by Lemma 10, cf. the second term on the right of (43).

So, to complete the proof we need to show that, for any $0 < \lambda_0 < \pi$,

$$\frac{1}{\widetilde{T}^{1/2}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\left(\left(\frac{2\pi}{\sigma^2}I_{\varepsilon,j} - 1\right) - H_{\theta_0,T}(j)\frac{1}{\widetilde{\widetilde{T}}}\sum_{k=j+1}^{\widetilde{T}}\gamma_{\theta_0,k}\left(\frac{2\pi}{\sigma^2}I_{\varepsilon,k} - 1\right)\right) \Rightarrow \frac{1}{\pi^{1/2}}B_\pi(\lambda), \tag{54}$$

in $[0,\lambda_0]$. Fidi's convergence follows by Lemma 7 part (b) after we note that the second term on the right of (53) is

$$\frac{1}{\widetilde{T}^{1/2}}\sum_{k=1}^{\widetilde{T}}\left(\frac{1}{\widetilde{T}}\sum_{j=1}^{k\wedge[\widetilde{T}\lambda/\pi]}H_{\theta_0,T}(j)\right)\gamma_{\theta_0,k}\left(\frac{2\pi}{\sigma^2}I_{\varepsilon,k}-1\right)$$

and $\left(\widetilde{T}^{-1}\sum_{j=1}^{k\wedge[\widetilde{T}\lambda/\pi]}H_{\theta_0,T}(j)\right)\gamma_{\theta_0,k}$ satisfies the same conditions of Lemma 7 for $\zeta(\lambda)$, *e.g.* those of $h_n(\lambda)$ in Giraitis, Hidalgo and Robinson's (2001) Theorem 4.2. Then, it suffices to prove tightness. Since α_T^0 is tight, we only need to show the tightness condition of

$$\Lambda_T(\lambda)=\frac{1}{\widetilde{T}}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_0,T}(j)\left(\frac{1}{\widetilde{T}^{1/2}}\sum_{k=j+1}^{\widetilde{T}}\gamma_{\theta_0,k}\left(I_{\varepsilon,k}-\frac{\sigma^2}{2\pi}\right)\right). \tag{55}$$

By Billingsley's (1968) Theorem 15.6, it suffices to show that

$$\mathbb{E}\left(\left|\Lambda_T(\vartheta)-\Lambda_T(\mu)\right|\left|\Lambda_T(\lambda)-\Lambda_T(\vartheta)\right|\right)\leq K\left|\lambda-\mu\right|^{2\delta}$$

for all $0\leq\mu<\vartheta<\lambda\leq\pi$ and some $\delta>1/2$. Observe that we can take $\widetilde{T}^{-1}<|\lambda-\mu|$ since otherwise the last inequality is trivial. Because $(\lambda-\vartheta)(\vartheta-\mu)<(\lambda-\mu)^2$, by the Cauchy-Schwarz's inequality, it suffices to show the last displayed inequality holds for $\mathbb{E}\left|\Lambda_T(\lambda)-\Lambda_T(\mu)\right|^2$ which is

$$\frac{1}{\widetilde{T}^3}\sum_{j,k=[\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_0,T}(j)\sum_{\ell_1=j+1}^{\widetilde{T}}\sum_{\ell_2=k+1}^{\widetilde{T}}\gamma_{\theta_0,\ell_1}\gamma'_{\theta_0,\ell_2}\mathbb{E}\left[\left(I_{\varepsilon,\ell_1}-\frac{\sigma^2}{2\pi}\right)\left(I_{\varepsilon,\ell_2}-\frac{\sigma^2}{2\pi}\right)\right]H'_{\theta_0,T}(k)$$

$$\leq\frac{K}{\widetilde{T}^2}\sum_{j,k=[\widetilde{T}\mu/\pi]+1}^{[\widetilde{T}\lambda/\pi]}\left\|H_{\theta_0,T}(j)\right\|\left\|H_{\theta_0,T}(k)\right\|\leq K\left(\left|\widetilde{H}(\lambda)-\widetilde{H}(\mu)\right|^2+\widetilde{T}^{-2}\log^2\widetilde{T}\right),$$

where $\widetilde{H}(\lambda):=\pi^{-1}\int_0^\lambda H_{\theta_0}(x)\,dx$ and $\left\|\widetilde{H}_T(\lambda)-\widetilde{H}(\lambda)\right\|\leq K\widetilde{T}^{-1}\log T$, where $\widetilde{H}_T(\lambda):=\widetilde{T}^{-1}\sum_{j=1}^{[\widetilde{T}\lambda/\pi]}\left\|H_{\theta_0,T}(j)\right\|$ by Lemma 1. From here we conclude by Billingsley's (1968) Theorem 15.6, because $\widetilde{H}(\lambda)$ is a monotonic, continuous and nondecreasing function such that $\left|\widetilde{H}(\lambda)-\widetilde{H}(\mu)\right|\leq K|\lambda-\mu|^\delta$, $\delta>1/2$ and $\widetilde{T}^{-1}\leq|\lambda-\mu|$. □

Proof of Theorem 4

By definition of $\beta_{\theta,T}$ and β_T^0, it suffices to show that

$$\left|\frac{1}{\widetilde{T}^{1/2}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}\left(\frac{I_{X,k}}{h_{\theta_T,k}}-I_{\varepsilon,k}\right)-H_{\theta_0,T}(k)\frac{1}{\widetilde{T}}\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_0,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-I_{\varepsilon,j}\right)\right| \tag{56}$$

and

$$\frac{1}{G_{\theta_T,T}(\pi)}\left(\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_0,T}(k)\frac{1}{\widetilde{T}^{1/2}}\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_0,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{G_{\theta_T,T}(\pi)}{2\pi}\right)\right)$$
$$-\frac{1}{G_{\theta_T,T}(\pi)}\left(\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_T,T}(k)\frac{1}{\widetilde{T}^{1/2}}\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_T,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{G_{\theta_T,T}(\pi)}{2\pi}\right)\right) \quad (57)$$

converge to zero uniformly in $\lambda\in[0,\pi]$. Expression (56) is $o_p(1)$, uniformly in $\lambda\in[0,\pi]$, because the contribution due to the term in brackets in the last line of (52), that is $-\phi'_{\theta_0,j}\left(2\pi\left(G_T^0(\pi)\right)^{-1}S_T^{-1}\widetilde{T}^{-1/2}\sum_{k=1}^{\tilde{T}}\phi_{\theta_0,k}I_{\varepsilon,k}\right)$ is easily seen to be zero. Next, because

$$\begin{aligned}&\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}\left\|\gamma_{\theta_0,k}\right\|\left\|A_{\theta_0,T}^{-1}(k)\right\|\frac{1}{\widetilde{T}}\sum_{j=k+1}^{\widetilde{T}}\left\|\gamma_{\theta_0,j}\right\|\\ \leq\;& K\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}\left\|\gamma_{\theta_0,k}\right\|\left\|A_{\theta_0,T}^{-1}(k)\left(1-\frac{k}{\widetilde{T}}\right)\right\|\\ \leq\;& K\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}\left\|\gamma_{\theta_0,k}\right\|\leq K\end{aligned}$$

by integrability of γ_{θ_0} and that $\left\|A_{\theta_0,T}(k)\left(1-k/\widetilde{T}\right)^{-1}\right\|>0$ by $A3$ and $A5$, it implies that the contribution into (56) due to the term $o_p(1)$ on the right of (52) is negligible.

Next we examine (57). Because (50) and (51), it suffices to show that

$$\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}\left\{\frac{H_{\theta_0,T}(k)}{\widetilde{T}^{1/2}}\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_0,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right)-\frac{H_{\theta_T,T}(k)}{\widetilde{T}^{1/2}}\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_T,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right)\right\} \quad (58)$$

converges to zero uniformly in $\lambda\in[0,\pi]$, after observing that

$$\sup_{\lambda\in[0,\pi]}\left|\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_T,T}(k)\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_T,j}-\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_0,T}(k)\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_0,j}\right|=0.$$

First, we observe that Lemmas 10 and 12 imply that it suffices to show the uniform convergence in $\lambda\in[0,\lambda_0]$ for any $\lambda_0<\pi$. But (58) is equal to

$$\frac{1}{\widetilde{T}}\sum_{k=1}^{[\widetilde{T}\lambda/\pi]}H_{\theta_T,T}(k)\frac{1}{\widetilde{T}^{1/2}}\sum_{j=k+1}^{\widetilde{T}}\left(\gamma_{\theta_0,j}-\gamma_{\theta_T,j}\right)\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right) \quad (59)$$

$$+\frac{1}{\widetilde{T}}\sum_{k=1}^{\left[\widetilde{T}\lambda/\pi\right]}\left(H_{\theta_0,T}\left(k\right)-H_{\theta_T,T}\left(k\right)\right)\frac{1}{\widetilde{T}^{1/2}}\sum_{j=k+1}^{\widetilde{T}}\gamma_{\theta_0,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right). \tag{60}$$

So, the theorem follows if (59) and (60) are $o_p(1)$ uniformly in $\lambda\in[0,\lambda_0]$.

To that end, we first show that

$$\sup_{\lambda\in[0,\pi]}\frac{1}{\widetilde{T}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}\left\|\phi_{\theta_0,j}-\phi_{\theta_T,j}\right\| = o_p\left(1\right), \tag{61}$$

$$\sup_{\lambda\in[0,\lambda_0]}\left\|A^{-1}_{\theta_0,T}\left(\lambda\right)-A^{-1}_{\theta_0}\left(\lambda\right)\right\| = o\left(1\right), \tag{62}$$

$$\sup_{\lambda\in[0,\lambda_0]}\left\|A^{-1}_{\theta_T,T}\left(\lambda\right)-A^{-1}_{\theta_0,T}\left(\lambda\right)\right\| = o_p\left(1\right). \tag{63}$$

(61) follows proceeding as with the proof of (44) in Lemma 11 but without the factor $h^{-1}_{\theta_T,j}I_{X,j}-\sigma^2/\left(2\pi\right)$, (62) follows because Assumption $A5$ implies that $A_{\theta_0}\left(\lambda_0\right)>0$ and because by Assumption $A3$ $\left\|\phi_{\theta_0}\left(\lambda\right)\phi'_{\theta_0}\left(\lambda\right)\right\|$ satisfies the same conditions of $\zeta\left(\lambda\right)$ in Lemma 1, so that

$$\sup_{\lambda\in[0,\lambda_0]}\left\|A_{\theta_0}\left(\lambda\right)-A_{\theta_0,T}\left(\lambda\right)\right\|=O\left(T^{-1}\log^2 T\right),$$

whereas (63) follows proceeding as with the proof of (61) and (62).

Now we show that (59) is $o_p(1)$ uniformly in $\lambda\in[0,\lambda_0]$, which follows by Lemma 11 and (61) – (63) noting that $\left(\gamma'_{\theta_0,j}-\gamma'_{\theta_T,j}\right)=\left(0,\phi'_{\theta_0,j}-\phi'_{\theta_T,j}\right)$, so does (60) by (61) and (63) and that

$$\sup_{\lambda\in[0,\pi]}\left|\frac{1}{\widetilde{T}^{1/2}}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\gamma_{\theta_0,j}\left(\frac{I_{X,j}}{h_{\theta_T,j}}-\frac{\sigma^2}{2\pi}\right)\right|=O_p\left(1\right)$$

by Lemmas 7 and 8 with $\zeta\left(\lambda\right)=\gamma_{\theta_0}\left(\lambda\right)$ there and observing (35) and that by Lemma 1, $\widetilde{T}^{-1}\sum_{j=\left[\widetilde{T}\lambda/\pi\right]+1}^{\widetilde{T}}\gamma_{\theta_0,j}\phi'_{\theta_0,j}\to\int_\lambda^\pi\gamma_{\theta_0}\left(x\right)\phi'_{\theta_0}\left(x\right)dx.$ □

Proof of Theorem 5

Under H_{1T}, we have that by definition,

$$\begin{aligned}G_{\theta_0,T}\left(\lambda\right) &= \frac{2\pi}{\widetilde{T}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}\frac{I_{X,j}}{h_j}+\frac{\sigma^2\tau}{\widetilde{T}^{3/2}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}l_j\\ &\quad+\frac{2\pi\tau}{\widetilde{T}^{3/2}}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}l_j\left(\frac{I_{X,j}}{h_j}-\frac{\sigma^2}{2\pi}\right)+\frac{1}{\widetilde{T}^2}\sum_{j=1}^{\left[\widetilde{T}\lambda/\pi\right]}s_{T,j}\frac{I_{X,j}}{h_j}.\end{aligned}$$

By Lemmas 1, 4 and 7 with $\zeta(\lambda) = \tau l(\lambda)$, and because $|s_T|$ is integrable, we have that

$$G_{\theta_0,T}(\lambda) = \frac{2\pi}{\widetilde{T}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} I_{\varepsilon,j} + \frac{\sigma^2 \tau}{\widetilde{T}^{1/2}\pi} \int_0^\lambda l(u)\, du + o_p\left(T^{-1/2}\right).$$

So, using (51) because $\int_0^\pi l(u)\, du = 0$, we have that uniformly in $\lambda \in [0,\pi]$,

$$\begin{aligned} \widetilde{T}^{1/2}\left(\frac{G_{\theta_0,T}(\lambda)}{G_{\theta_0,T}(\pi)} - \frac{\lambda}{\pi}\right) &= \widetilde{T}^{1/2}\left(\frac{2\pi}{G_T^0(\pi)\,\widetilde{T}} \sum_{j=1}^{[\widetilde{T}\lambda/\pi]} I_{\varepsilon,j} - \frac{\lambda}{\pi} + \frac{\tau}{\widetilde{T}^{1/2}\pi} \int_0^\lambda l(u)\, du\right) \\ &\quad + o_p(1) \\ &= \alpha_T^0(\lambda) + \frac{\tau}{\pi} \int_0^\lambda l(u)\, du + o_p(1). \end{aligned}$$

From here the conclusion is straightforward. □

References

Aki, S. (1986). Some test statistics based on the martingale term of the empirical distribution function. *Ann. Inst. Statist. Math.* **38** 1-21.

Anderson, T.W. (1997). Goodness-of-fit for autoregressive processes. *J. Time Ser. Anal.* **18** 321-339.

Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

Bloomfield, P. (1973). An exponential model for the spectrum of a scalar time series. *Biometrika* **60** 217-226.

Box, G.E.P. and Pierce, D.A. (1970). Distribution of residual autocorrelations in autoregressive-integrated moving average time series models. *J. Amer. Statist. Assoc.* **65** 1509-1526.

Brillinger, D.R. (1981). *Time Series, Data Analysis and Theory.* Holden-Day, San Francisco.

Brockwell, P.J. and Davis, R.A. (1991). *Time series: Theory and methods.* Springer-Verlag, New York.

BROWN, R.L., DURBIN, J. AND EVANS, J.M. (1975). Techniques for testing constancy of regression relationships over time (with discussion). *J. Roy. Statist. Soc. Ser. B.* **37** 149-192.

CHEN, H. AND ROMANO J.P. (1999). Bootstrap-assisted goodness-of-fit tests in the frequency domain. *J. Time Ser. Anal.* **20** 619-654.

DAHLHAUS, R. (1985). On the asymptotic distribution of Bartlett's U_p-statistic. *J. Time Ser. Anal.* **6** 213-227.

DELGADO, M.A. AND J. HIDALGO (2000). Bootstrap goodness-of-fit test for linear processes. Preprint.

DURBIN, J., KNOTT, M. AND TAYLOR, C.C. (1975). Components of Cramér-von Mises statistics II. *J. Roy. Statist. Soc. Ser. B.* **37** 216-237.

EUBANK, R.L. AND LARICCA, V. (1992): Asymptotic comparison of Cramér-von Mises and nonparametric techniques for testing goodness-of-fit. *Ann. Statist.* **20** 2071-2086.

GIRAITIS, L., HIDALGO, J. AND ROBINSON, P.M. (2001). Gaussian estimation of parametric spectral density with unknown pole. *Ann. Statist.* **29** 987-1023.

GIRAITIS, L. AND SURGAILIS, D. (1990). A central limit theorem for quadratic forms in strongly dependent linear variables and its applications to asymptotic normality of Whittle's estimates. *Probab. Theory Related Fields* **86** 87-104.

GRENANDER, U. (1950). Stochastic processes and statistical inference. *Ark. Mat.* **1** 195-277.

GRENANDER, U. (1981). *Abstract Inference.* John Wiley, New York.

GRENANDER, U. AND ROSENBLATT, M. (1957). *Statistical Analysis of Stationary Time Series.* John Wiley, New York.

HAINZ, G. AND DAHLHAUS, R. (2000). Spectral domain bootstrap tests for stationary time series. Preprint.

Hannan, E.J. (1973). The asymptotic theory of linear time series models. *J. Appl. Probab.* **10** 130-145.

Hong, Y. (1996). Consistent testing for serial-correlation of unknown form. *Econometrica* **64** 837-864.

Hosking, J.R.M. (1984). Modeling persistence in hydrological time series using fractional differencing. *Water Resources Research* **20** 1898-1908.

Hosoya, Y. (1997). A limit theory for long-range dependence and statistical inference on related models. *Ann. Statist.* **25** 105-137.

Kac, M. and Siegert, A.J.F. (1947). An explicit representation of a stationary Gaussian process *Ann. Math. Statist.* **18** 438-442.

Khmaladze, E.V. (1981). Martingale approach in the theory of Goodness-of-fit tests. *Theory Probab. Appl.* **26** 240-257.

Khmaladze, E.V. and Koul, H. (2004). Martingale transforms goodness-of-fit tests in regression models. *Ann. Statist.* **32** 995-1034.

Klüppelberg, C. and Mikosch, T. (1996). The integrated periodogram for stable processes. *Ann. Statist.* **24** 1855-1879.

Koul, H. and Stute, W. (1998). Regression model fitting with long memory errors. *J. Statist. Plann. Inference* **71** 35-56.

Koul, H. and Stute, W. (1999). Nonparametric model checks in time series. *Ann. Statist.* **27** 204-237.

Ljung, G.M. and Box G.E.P. (1978). On a measure of lack of fit in time series models. *Biometrika* **65** 297-303.

Neyman, J. (1937). "Smooth" test for goodness of fit. *Skan. Aktuarietidskr.* **20** 150-199.

Nikabadze, A. and Stute, W. (1997). Model checks under random censorship. *Statist. Probab. Lett.* **32** 249-259.

Paparoditis, E. (2000). Spectral density based goodness-of-fit tests for time series models," *Scand. J. Statist.* **27** 143-176.

Prewitt, K. (1998). Goodness-of-fit test in parametric time series models differenced, *J. Time Ser. Anal.* **19** 549-574.

Robinson, P.M. (1994). Time Series with strong dependence. In C.A. Sims (ed.), *Advances in Econometrics: Sixth World Congress*, vol. 1, pp. 47-95. Cambridge: Cambridge University Press.

Robinson, P.M. (1995a). Log-periodogram regression for time series with long range dependence. *Ann. Statist.* **23** 1048-1072.

Robinson, P.M. (1995b). Gaussian semiparametric estimation of long-range dependence. *Ann. Statist.* **23** 1630-1661.

Schoenfeld, D.A. (1977). Asymptotic properties of tests based on linear combinations of the Cramér-von Mises statistic. *Ann. Statist.* **5** 1017-1026.

Sen, P.K. (1982). Invariance principles for recursive residuals. *Ann. Statist.* **10** 307-312.

Shorack, G.R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics.* John Wiley, New York.

Stute, W. (1997). Nonparametric model checks for regression. *Ann. Statist.* **25** 613-641.

Stute, W., Thies, S. and Zhu, L. (1998). Model checks for regression: An innovation process approach. *Ann. Statist.* **26** 1916-1934.

Stute, W. and Zhu, L. (2002). Model checks for generalized linear processes. *Scand. J. Statist.* **29** 535-545.

Velasco, C. (1999). Non-stationary log-periodogram regression *J. Econometrics* **91** 325-371.

Velasco, C. and Robinson, P.M. (2000). Whittle pseudo-maximum likelihood estimates for non-stationary time series. *J. Amer. Statist. Assoc.* **95** 1229-1243.

VELILLA, S. (1994). A goodness-of-fit test for autoregressive-moving-average models based on the standardized sample distribution of the residuals *J. Time Ser. Anal.* **15** 637-648.

Table 1.

Empirical size of omnibus and Portmanteau tests at 5% of significance.

	$T=200$						$T=500$					
	$\hat{C}_T$	C_T^0	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{10,T}$	$\tilde{Q}_{20,T}$	$\hat{C}_T$	C_T^0	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{15,T}$	$\tilde{Q}_{35,T}$
$\delta_0,\ H_0:\mathrm{AR}(1)$												
-0.8	4.92	4.69	3.34	3.72	3.91	3.61	5.07	5.17	3.56	3.87	4.35	3.97
-0.5	4.38	4.96	2.80	3.38	3.60	3.41	4.96	5.16	3.12	3.75	4.17	3.82
0.0	4.07	4.96	2.66	3.35	3.45	3.37	4.62	5.10	3.00	3.63	4.11	3.82
0.5	3.59	4.95	2.67	3.33	3.57	3.40	4.50	5.04	2.97	3.82	4.17	3.80
0.8	3.08	4.92	2.89	3.44	3.73	3.54	4.27	5.11	3.33	3.77	4.32	3.88
$\eta_0,\ H_0:\mathrm{MA}(1)$												
-0.8	4.25	8.37	4.32	4.54	4.42	3.95	4.89	6.67	4.13	4.39	4.56	4.07
-0.5	4.16	5.06	2.83	3.41	3.65	3.38	4.89	5.18	3.13	3.76	4.15	3.83
0.0	4.08	4.96	2.51	3.26	3.46	3.32	4.62	5.10	2.94	3.61	4.05	3.82
0.5	3.60	5.08	2.65	3.30	3.55	3.41	4.49	5.15	2.96	3.77	4.13	3.82
0.8	3.89	7.72	15.33	15.30	15.33	15.05	4.63	6.42	8.03	8.44	8.68	8.17
$d_0,\ H_0:\mathrm{I}(d)$												
0.0	3.53	4.96	2.76	3.40	3.68	3.47	4.48	5.10	3.13	3.90	4.29	3.83
0.2	3.54	4.95	2.76	3.39	3.63	3.46	4.54	5.15	3.14	3.89	4.27	3.81
0.4	3.58	5.21	2.79	3.39	3.59	3.44	4.58	5.37	3.14	3.88	4.27	3.80

Table 2.

Empirical power of omnibus and Portmanteau tests at 5% of significance.

H_0 : AR(1). H_1 : MA(1).

	$T=200$					$T=500$				
η	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{10,T}$	$\tilde{Q}_{20,T}$	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{15,T}$	$\tilde{Q}_{35,T}$
-0.8	100.00	99.97	99.95	99.25	92.34	100.00	100.00	100.00	100.00	100.00
-0.5	80.82	70.16	55.53	44.38	31.25	99.84	99.23	97.54	88.65	68.72
0.2	7.12	5.04	4.98	4.86	4.34	12.16	8.31	7.35	6.27	5.21
0.5	70.82	72.03	57.50	46.06	32.15	98.59	99.32	97.83	89.19	69.29
0.8	99.56	99.99	99.95	99.30	92.76	100.00	100.00	100.00	100.00	100.00

H_0 : MA(1). H_1 : AR(1).

	$T=200$					$T=500$				
δ	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{10,T}$	$\tilde{Q}_{20,T}$	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{15,T}$	$\tilde{Q}_{35,T}$
-0.8	100.00	100.00	100.00	100.00	99.99	100.00	100.00	100.00	100.00	100.00
-0.5	84.36	77.15	66.51	57.37	44.02	99.73	99.47	98.45	94.26	82.89
0.2	7.16	3.71	3.99	3.94	3.63	12.04	6.65	6.42	5.73	4.80
0.5	77.08	74.86	64.04	54.79	31.78	99.19	99.41	98.35	93.77	82.04
0.8	100.00	100.00	100.00	100.00	99.97	100.00	100.00	100.00	100.00	100.00

H_0 : I(d). H_1 : AR(1).

	$T=200$					$T=500$				
δ	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{10,T}$	$\tilde{Q}_{20,T}$	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{15,T}$	$\tilde{Q}_{35,T}$
0.2	11.34	12.84	13.00	11.27	13.13	34.92	33.35	33.01	23.98	15.71
0.5	26.81	34.11	41.17	35.55	24.94	75.29	81.36	87.81	80.73	58.52
0.8	9.82	12.86	21.01	21.32	15.41	33.21	38.74	57.53	61.63	39.15

H_0 : AR(1). H_1 : I(d).

	$T=200$					$T=500$				
d	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{10,T}$	$\tilde{Q}_{20,T}$	$\hat{C}_T$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$	$\tilde{Q}_{15,T}$	$\tilde{Q}_{35,T}$
0.1	8.22	4.98	5.66	5.11	4.83	16.79	12.07	14.09	12.34	9.10
0.2	19.90	13.74	16.20	15.23	11.81	51.77	45.04	53.29	47.54	36.11
0.3	36.03	25.92	32.00	30.50	24.35	82.80	74.84	85.12	81.44	69.62
0.4	48.83	34.86	43.78	43.31	35.48	94.40	87.30	95.56	94.31	87.38

Table 3.

Empirical size and power under local alternatives at 5% of significance

H_0 : AR(1). H_1 : ARFIMA(1,$d=\tau/T^{1/2}$,0).

$T=200$										
τ	ρ	$\hat{C}_T$	$\hat{W}_{3,T}$	$\hat{W}_{6,T}$	$\left\|\hat{\psi}_{3,T}\right\|$	$\left\|\hat{\psi}_{6,T}\right\|$	$\hat{\psi}^+_{3,T}$	$\hat{\psi}^+_{6,T}$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$
0	0.0	4.07	3.19	2.59	4.70	4.81	4.48	5.12	2.66	3.35
	0.5	3.59	2.98	2.32	3.79	4.24	3.62	3.99	2.67	3.33
	0.8	3.08	2.52	1.94	3.94	3.10	3.75	4.02	2.89	3.44
1	0.0	6.26	5.40	4.37	8.39	11.13	13.44	16.63	3.68	4.25
	0.5	3.57	2.90	2.26	3.45	4.19	4.19	5.64	2.73	3.37
	0.8	3.01	2.25	1.66	4.10	4.52	7.80	8.53	3.87	4.41
2	0.0	12.19	12.04	10.53	19.93	26.15	28.94	35.10	7.80	9.13
	0.5	3.44	2.91	2.36	3.47	4.15	4.25	6.27	2.91	3.58
	0.8	4.84	3.16	2.19	9.17	10.33	16.59	17.98	8.45	7.58
3	0.0	21.92	23.63	21.27	35.77	44.37	47.20	54.61	15.17	18.02
	0.5	3.26	2.74	2.39	3.65	4.43	4.99	6.48	3.27	3.92
	0.8	9.13	6.61	4.10	20.13	22.90	31.95	35.14	21.18	16.12
4	0.0	33.38	27.13	24.15	50.40	59.39	62.18	69.12	23.88	29.88
	0.5	3.41	2.47	2.38	4.09	4.75	6.80	7.61	4.32	4.67
	0.8	17.48	14.65	9.09	38.10	43.37	53.13	57.56	46.00	33.97

$T=500$										
τ	ρ	$\hat{C}_T$	$\hat{W}_{3,T}$	$\hat{W}_{6,T}$	$\left\|\hat{\psi}_{3,T}\right\|$	$\left\|\hat{\psi}_{6,T}\right\|$	$\hat{\psi}^+_{3,T}$	$\hat{\psi}^+_{6,T}$	$\hat{Q}_{3,T}$	$\hat{Q}_{6,T}$
0	0.0	4.62	4.22	3.66	4.81	4.78	4.57	5.06	3.00	3.63
	0.5	4.50	3.99	3.40	4.26	4.58	4.27	4.43	2.97	3.82
	0.8	4.27	3.56	3.09	3.90	3.85	4.63	3.63	3.33	3.77
1	0.0	6.93	7.03	6.29	9.35	11.62	14.63	17.54	4.37	5.13
	0.5	4.58	4.42	4.08	4.85	5.35	58.3	7.43	3.02	3.93
	0.8	4.74	4.13	3.47	5.72	5.90	9.61	9.83	4.12	4.64
2	0.0	14.22	15.51	14.23	23.43	29.37	33.47	39.37	10.03	11.60
	0.5	4.69	4.72	4.67	4.83	6.49	6.37	10.18	3.08	4.21
	0.8	7.36	6.13	4.73	11.57	12.08	19.11	19.81	7.27	7.38
3	0.0	26.86	31.03	29.55	44.70	53.35	56.44	63.59	21.28	24.91
	0.5	4.65	5.04	5.48	4.71	7.14	5.44	11.31	3.30	4.60
	0.8	13.56	11.62	8.18	23.46	24.65	34.56	35.78	15.23	13.51
4	0.0	43.62	51.19	49.81	66.34	74.28	75.93	81.84	37.13	43.93
	0.5	4.65	5.18	6.35	5.05	7.03	5.09	10.80	3.81	5.09
	0.8	24.44	23.10	16.17	42.07	44.05	54.86	56.23	31.28	25.74

$\left|\hat{\psi}_{n,T}\right|$ denotes two sided tests, whereas $\hat{\psi}^+_{n,T}$ are one sided (right hand side) tests.

Table 4.

Empirical size and power under local alternatives at 5% of significance

H_0 : AR(1). H_1 : ARMA(1,1), $\psi = \tau / T^{1/2}$.

		$T = 200$								
τ	ρ	$\hat{C}_T$	$\hat{W}_{3,T}$	$\hat{W}_{6,T}$	$\lvert\hat{\psi}_{3,T}\rvert$	$\lvert\hat{\psi}_{6,T}\rvert$	$\hat{\psi}^+_{3,T}$	$\hat{\psi}^+_{6,T}$	$\tilde{Q}_{3,T}$	$\tilde{Q}_{6,T}$
0	0.0	4.13	3.09	3.58	3.98	4.39	4.18	4.39	2.65	3.36
	0.5	3.62	2.80	2.22	3.68	4.04	3.93	4.14	2.67	3.31
	0.8	3.06	2.38	1.86	3.00	3.21	3.45	3.64	2.93	3.46
1	0.0	4.22	3.10	2.58	3.88	4.23	3.74	3.93	2.76	3.40
	0.5	5.52	4.08	2.90	5.51	5.76	8.86	9.20	3.08	3.61
	0.8	7.81	5.63	3.66	7.77	7.98	13.13	13.62	5.47	5.05
2	0.0	5.01	3.50	2.79	3.77	4.06	3.36	3.46	3.45	3.82
	0.5	8.53	6.10	4.02	8.58	9.06	14.33	14.61	4.51	4.56
	0.8	18.07	13.73	8.53	20.63	21.26	30.93	31.41	12.52	10.66
3	0.0	7.79	5.04	3.76	4.62	4.92	6.00	6.06	5.60	5.32
	0.5	10.64	7.80	5.16	10.84	11.25	17.39	17.87	5.76	5.41
	0.8	32.10	27.17	17.65	37.68	38.18	50.25	50.49	23.84	20.09
4	0.0	14.60	9.51	6.65	10.86	11.01	16.70	16.78	11.03	8.99
	0.5	10.67	8.16	5.42	10.65	11.01	17.11	17.57	5.93	5.56
	0.8	45.29	42.62	29.55	52.48	52.79	64.96	64.97	36.18	31.63

		$T = 500$								
τ	ρ	$\hat{C}_T$	$\hat{W}_{3,T}$	$\hat{W}_{6,T}$	$\lvert\hat{\psi}_{3,T}\rvert$	$\lvert\hat{\psi}_{6,T}\rvert$	$\hat{\psi}^+_{3,T}$	$\hat{\psi}^+_{6,T}$	$\hat{Q}_{3,T}$	$\hat{Q}_{6,T}$
0	0.0	4.70	4.43	3.86	4.66	5.68	4.52	4.62	2.99	3.64
	0.5	4.50	4.23	3.70	4.53	4.55	4.50	4.52	2.99	3.80
	0.8	4.39	3.94	3.40	4.22	4.26	4.37	4.38	3.34	3.78
1	0.0	4.74	4.37	3.83	4.70	4.75	4.31	4.35	3.02	3.70
	0.5	6.68	5.72	4.73	6.71	6.61	10.25	10.36	3.75	4.32
	0.8	9.56	8.06	6.00	10.03	10.08	16.20	16.28	6.26	5.82
2	0.0	5.00	4.47	3.90	4.76	4.87	3.61	3.62	3.34	3.90
	0.5	11.06	8.94	6.81	11.48	11.43	18.23	18.17	6.06	5.88
	0.8	23.21	19.66	13.89	26.87	26.88	38.01	37.99	15.66	13.35
3	0.0	6.31	5.17	4.38	4.95	5.03	3.19	3.18	4.25	4.55
	0.5	16.44	13.17	9.58	17.26	17.24	26.26	26.03	9.45	8.39
	0.8	42.78	38.92	28.30	50.11	49.91	62.36	62.42	32.23	27.37
4	0.0	9.48	6.98	5.57	5.09	5.16	4.09	4.07	6.40	5.98
	0.5	21.08	17.22	12.42	22.10	21.95	32.15	31.99	12.84	10.89
	0.8	62.44	60.69	47.41	70.99	70.86	80.69	80.67	52.01	46.42

$\left|\hat{\psi}_{n,T}\right|$ denotes two sided tests, whereas $\hat{\psi}^+_{n,T}$ are one sided (right hand side) tests.